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Research Article

Existence of a Unique Solution and the Hyers–Ulam-H-Fox Stability of the Conformable Fractional Differential Equation by Matrix-Valued Fuzzy Controllers

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In this paper, we consider a conformable fractional differential equation with a constant coefficient and obtain an approximation for this equation using the Radu–Mihet method, which is derived from the alternative fixed- point theorem. Considering the matrix-valued fuzzy k-normed spaces and matrix-valued fuzzy H-Fox function as a control function, we investigate the existence of a unique solution and Hyers–Ulam-H-Fox stability for this equation. Finally, by providing numerical examples, we show the application of the obtained results.

1. Introduction

One of the important topics in mathematics and especially in mathematical analysis is fractional calculus. Here, we can refer to the fractional derivatives of Caputo, Riemann–Liouville, Grunwald–Letnikov, Marchaud, or Hadamard as fractional operators. It should be noted that these operators are the result of changes made to ODEs and PDEs over time. Various kinds of fractional derivatives have been discussed by Kilbas in [1] and Butzer et al. in [2]. Derivatives such as the Caputo, Riemann–Liouville, or Hadamard fractional derivatives have complex rules such as the law of chain. The researchers decided to find another derivative to get rid of these complications. Thus, a local fractional derivative containing a limit was proposed instead of a single integral called the consistent fractional derivative. These derivatives have many uses and properties. For example, they are used to extend Newton mechanics [3–6]. Researchers have recently introduced a new type of derivative that modifies conformable fractional derivatives. They have also studied the method change of the parameters for the conformable fractional differential equations by considering a regular fractional generalization of the Sturm–Liouville eigenvalue problem [7–9].

In this paper, we consider an MVkFB-space introduced in [10] and consider a modern class of the MVF control function based on the H-Fox functions. Our goal is to obtain an approximation for the conformable fractional differential equation using the alternative fixed-point theorem in MVkFB-spaces. The fuzzy control functions presented in this paper have a dynamic situation and can model new events, such as the COVID-19 disease, as explained in [11]. Using fuzzy controllers, the stability analysis of differential equations and integral equations can be studied.

We consider the following conformable FDE with constant coefficients:

$$\begin{cases} \mathfrak{D}_\omega^a \mathfrak{f}(\varpi - \vartheta) = \varrho \mathfrak{f}(\varpi - \vartheta) + K(\varpi - \vartheta, \mathfrak{f}(\varpi - \vartheta)), & \varpi, \vartheta \in (a, b], 0 < \omega < 1, \\ \mathfrak{f}(a) = \mathfrak{f}_a, \end{cases} \tag{1}$$

where $\mathfrak{D}_\omega^a \mathfrak{f}$ is called the conformable fractional derivative (CFD) with a lower index a of the function $\mathfrak{f}$ and $\mathcal{T} = [a, b]$, $K \in C(\mathcal{T} \times \mathbb{R}, \mathbb{R})$[1, 2].

The paper is organized as follows: In the second section, we present the basic definitions and concepts that are necessary to investigate the main results, and we also introduce the matrix-valued fuzzy H-Fox function as a control function. In the third section, using the alternative FPT, we prove the existence of a unique solution and the Hyers–Ulam-H-Fox stability for the conformable FDE in MVFkN-spaces, and at the end, as an application, we provide a numerical example.

2. Preliminaries

Definition 1. For a mapping $\mathfrak{f}: [a, \infty] \longrightarrow \mathbb{R}$, the CFD starting from a of order ω is defined by

$$\mathfrak{D}_\omega^a \mathfrak{f}(\varpi - \vartheta) = \lim_{i \longrightarrow 0} \frac{\mathfrak{f}\left((\varpi - \vartheta) + i((\varpi - \vartheta) - a)^{1-\omega}\right) - \mathfrak{f}(\varpi - \vartheta)}{i}, (\varpi - \vartheta) > a, 0 < \omega < 1. \tag{2}$$

If on $(a, b) \mathfrak{D}_\omega^a \mathfrak{f}(\varpi - \vartheta)$ exists, then $\mathfrak{D}_\omega^a \mathfrak{f}(a) = \lim_{(\varpi - \vartheta) \longrightarrow a^+} \mathfrak{D}_\omega^a \mathfrak{f}(\varpi - \vartheta)$.

Remark 1. For a finite given $\mathfrak{D}_\omega^a \mathfrak{f}((\varpi - \vartheta)_0)$, $\mathfrak{f}$ is ω–differentiable at $(\varpi - \vartheta)_0$. If $\mathfrak{f} \in C^1([a, \infty), \mathbb{R})$, then $\mathfrak{D}_\omega^a \mathfrak{f}(\varpi - \vartheta) = ((\varpi - \vartheta) - a)^{1-\omega} \mathfrak{f}'(\varpi - \vartheta)$.

Definition 2. For a mapping $\mathfrak{f}(\varpi - \vartheta)$, the Hadamard fractional integral with the order $0 < \omega < 1$ and parameter $s \in \mathbb{R}$ is defined by

$${}_a^H I_{(\varpi - \vartheta)}^{\omega, s} \mathfrak{f}(\varpi - \vartheta) = \frac{1}{\Gamma(\omega)} \int_a^{\varpi - \vartheta} \left(\frac{\lambda}{t}\right)^s \left(\log \frac{t}{\lambda}\right)^{\omega - 1} \frac{\mathfrak{f}(\lambda)}{\lambda} d\lambda, \tag{3}$$

where $(\varpi - \vartheta) \in (a, b)$ and $a \leq b$ in $\mathbb{R}$.

Lemma 1. *Let $\mathfrak{f} \in C^1([a, \infty))$. For the real-valued mapping $\mathfrak{f}$ and $(\varpi - \vartheta) > a$, $0 < \omega < 1$, the following relationship is always established:*

$${}_a^H I_{(\varpi - \vartheta)}^{\omega, s} \mathfrak{D}_\omega^a \mathfrak{f}(\varpi - \vartheta) = \mathfrak{f}(\varpi - \vartheta) - \mathfrak{f}(a). \tag{4}$$

Theorem 1. *By considering the Mittag-Leffler map, we obtain the following equation:*

$$E_\tau\left(\left(\frac{((\varpi - \vartheta) - a)^\omega}{\omega}\right)^\varrho\right) = \sum_{\varsigma=0}^{\infty} \frac{\left(((\varpi - \vartheta) - a)^\omega / \omega\right)^{\varsigma\varrho}}{\Gamma(1 + \tau\varsigma\varrho)}, \quad \tau > 0. \tag{5}$$

Suppose that $\mathcal{Z}(\varpi - \vartheta) = E_\tau((((\varpi - \vartheta) - a)^\omega / \omega)^\varrho)$, then we obtain the following equation:

$$\mathfrak{D}_\omega^a \mathcal{Z}(\varpi - \vartheta) = \varrho \mathcal{Z}(\varpi - \vartheta). \tag{6}$$

Proof. Using Remark 1, we have

$$\begin{aligned} \mathfrak{D}_\omega^a \mathcal{Z}(\varpi - \vartheta) &= ((\varpi - \vartheta) - a)^{1-\omega} \frac{(\varpi - \vartheta)\mathcal{Z}(\varpi - \vartheta)}{b(\varpi - \vartheta)} \\ &= ((\varpi - \vartheta) - a)^{1-\omega} \varrho ((\varpi - \vartheta) - a)^{\omega - 1} \\ &\quad \cdot \sum_{\varsigma=0}^{\infty} \frac{\left(((\varpi - \vartheta) - a)^\omega / \omega\right)^{\varsigma\varrho}}{\Gamma(1 + \tau\varsigma\varrho)} \\ &= \lambda \sum_{\varsigma=0}^{\infty} \frac{\left(((\varpi - \vartheta) - a)^\omega / \omega\right)^{\varsigma\varrho}}{\Gamma(1 + \tau\varsigma\varrho)} \\ &= \varrho \mathcal{Z}(\varpi - \vartheta). \end{aligned} \tag{7}$$

Next, we study the mapping $\mathfrak{f}$. □

Theorem 2. *If for equation (1), the mapping $\mathfrak{f} \in C(\mathcal{T}, \mathbb{R})$ is a solution, thus we have*

$$\begin{aligned} \mathfrak{f}(\varpi - \vartheta) &= \mathfrak{f}_a \sum_{\varsigma=0}^{\infty} \frac{\left(((\varpi - \vartheta) - a)^\omega / \omega\right)^{\varsigma\varrho}}{\Gamma(1 + \tau\varsigma\varrho)} \\ &\quad + \frac{1}{\Gamma(\omega)} \int_a^{\varpi - \vartheta} \sum_{\varsigma=0}^{\infty} \frac{\left(((\varpi - \vartheta) - a)^\omega / \omega\right)^{\varsigma\varrho}}{\Gamma(1 + \tau\varsigma\varrho)} \\ &\quad \cdot \sum_{\varsigma=0}^{\infty} \frac{\left((\lambda - a)^\omega / \omega\right)^{-\varsigma\varrho}}{\Gamma(1 + \tau\varsigma\varrho)} \left(\frac{\lambda}{t}\right)^s \left(\log \frac{t}{\lambda}\right)^{\omega - 1} \frac{K(\lambda, \mathfrak{f}(\lambda))}{\lambda} d\lambda. \end{aligned} \tag{8}$$

Proof. For any solution of (1), it should be as follows:

$$\mathfrak{f}(\varpi-\vartheta)=\sum_{\varsigma=0}^{\infty}\frac{\left(((\varpi-\vartheta)-a)^{\omega}/\omega\right)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}c(\varpi-\vartheta), \quad (9)$$

where $c(\varpi-\vartheta)$ is an unknown continuously differentiable function. From (9) and Remark 1, we get

$$\begin{aligned}\mathfrak{D}_{\omega}^{a}\mathfrak{f}(\varpi-\vartheta)&=\mathfrak{D}_{\omega}^{a}\left(\sum_{\varsigma=0}^{\infty}\frac{\left((\varpi-\vartheta)-a^{\omega}/\omega\right)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}c(\varpi-\vartheta)\right)\\&=((\varpi-\vartheta)-a)^{1-\omega}\sum_{\varsigma=0}^{\infty}\frac{\left(((\varpi-\vartheta)-a)^{\omega}/\omega\right)^{k\varrho}}{\Gamma(1+\tau\varsigma\varrho)}((\varpi-\vartheta)-a)^{\omega-1}c(\varpi-\vartheta)\\&\quad+\sum_{\varsigma=0}^{\infty}\frac{\left(((\varpi-\vartheta)-a)^{\omega}/\omega\right)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}((\varpi-\vartheta)-a)^{1-\omega}c'(\varpi-\vartheta)\\&=\varrho c(\varpi-\vartheta)\sum_{\varsigma=0}^{\infty}\frac{\left(((\varpi-\vartheta)-a)^{\omega}/\omega\right)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}+\sum_{\varsigma=0}^{\infty}\frac{\left(((\varpi-\vartheta)-a)^{\omega}/\omega\right)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\mathfrak{D}_{\omega}^{a}c(\varpi-\vartheta)\\&=\varrho\mathfrak{f}(\varpi-\vartheta)+\sum_{\varsigma=0}^{\infty}\frac{\left(((\varpi-\vartheta)-a)^{\omega}/\omega\right)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\mathfrak{D}_{\omega}^{a}c(\varpi-\vartheta).\end{aligned} \quad (10)$$

As a result, we obtain the following equation:

$$\mathfrak{D}_{\omega}^{a}c(\varpi-\vartheta)=\sum_{\varsigma=0}^{\infty}\frac{\left((((\varpi-\vartheta)-a)^{\omega})/\omega\right)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}K(\varpi-\vartheta,\mathfrak{f}(\varpi-\vartheta)). \quad (11)$$

From (11) and Lemma 1, we get

$$c(\varpi-\vartheta)=c(a)+\frac{1}{\Gamma(\omega)}\int_{\mathrm{a}}^{\varpi-\vartheta}\sum_{\varsigma=0}^{\infty}\frac{\left((\lambda-\mathfrak{c}c)^{\omega}/\omega\right)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\left(\frac{\lambda}{\mathrm{t}}\right)^{s}\left(\log\frac{\mathrm{t}}{\lambda}\right)^{\omega-1}\frac{K(\lambda,\mathfrak{f}(\lambda))}{\lambda}d\lambda, \quad (12)$$

where $c(a)=\mathfrak{f}(a)=\mathfrak{f}_a$.

By using (9) and (12), the desired result is obtained. □ □

Definition 3. A mapping $\mathfrak{f}\in C^1(\mathcal{T},\mathbb{R})$ is said to be the solution of (1) if $\mathfrak{f}$ satisfies $\mathfrak{D}_{\omega}^{a}\mathfrak{f}(\varpi-\vartheta)=\varrho\mathfrak{f}(\varpi-\vartheta)+K(\varpi-\vartheta,\mathfrak{f}(\varpi-\vartheta))$, $\varpi-\vartheta\in(a,\mathfrak{d}]$ and $\mathfrak{f}(a)=\mathfrak{f}_a$. Thus, we obtain the following equation:

$$\begin{aligned}\mathfrak{f}(\varpi-\vartheta)&=\mathfrak{f}_a\sum_{\varsigma=0}^{\infty}\frac{\left(((\varpi-\vartheta)-a)^{\omega}/\omega\right)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\\&\quad+\frac{1}{\Gamma(\omega)}\int_{a}^{\varpi-\vartheta}\sum_{\varsigma=0}^{\infty}\frac{\left(((\varpi-\vartheta)-a)^{\omega}/\omega\right)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\\&\quad\cdot\sum_{\varsigma=0}^{\infty}\frac{\left((\lambda-a)^{\omega}/\omega\right)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\left(\frac{\lambda}{\mathrm{t}}\right)^{s}\left(\log\frac{\mathrm{t}}{\lambda}\right)^{\omega-1}\frac{\mathrm{K}(\lambda,\mathfrak{f}(\lambda))}{\lambda}d\lambda.\end{aligned} \quad (13)$$

Definition 4 (see [12]). The multivariate Mittag-Leffler (MM-L) function is defined by the following series representation:

$$\begin{aligned}&E_{(\alpha_1,\ldots,\alpha_m),\beta}(u_1,\ldots,u_m)\\&=\sum_{\varsigma=0}^{\infty}\sum_{\varsigma_1+\cdots+\varsigma_m=\varsigma}\binom{\varsigma}{\varsigma_1,\ldots,\varsigma_m}\frac{u_1^{\varsigma_1}\cdots u_m^{\varsigma_m}}{\Gamma(\alpha_1\varsigma_1+\cdots+\alpha_m\varsigma_m+\beta)},\end{aligned} \quad (14)$$

where $\alpha_i,\beta>0$ for $i=1,2,\ldots,m$.

Definition 5 (see [13–15]). According to a standard notation, the Fox $\mathcal{H}$ function is defined as

$$\mathcal{H}_{\tau,\omega}^{m,n}(\mathfrak{g})=\frac{1}{2\pi i}\int_{\mathcal{L}}\mathcal{H}_{\tau,\omega}^{m,n}(e)\mathfrak{g}^{e}de, \quad (15)$$

where $\mathcal{L}$ is a suitable path in the complex plane $\mathbb{C}$ to be disposed later. $\mathfrak{g}^{e}=\exp\{(\log|\mathfrak{g}|+i\arg\mathfrak{g})\}$ and

$$\mathcal{H}^{m,n}_{\tau,\omega}(e) = \frac{\mathbb{V}(e)\mathbb{W}(e)}{\mathbb{X}(e)\mathbb{Y}(e)}, \quad (16)$$

$$\mathbb{V}(e) = \prod_{j=1}^{m} \Gamma\left(s_j - \psi_j e\right),$$
$$\mathbb{W}(e) = \prod_{j=1}^{n} \Gamma\left(1 - r_j + \phi_j e\right), \quad (17)$$

$$\mathbb{X}(e) = \prod_{j=m+1}^{q} \Gamma\left(1 - s_j + \psi_j e\right),$$
$$\mathbb{Y}(e) = \prod_{j=n+1}^{p} \Gamma\left(r_j - \phi_j e\right), \quad (18)$$

with $0 \le n \le p$, $1 \le m \le q$, $\{r_j, s_j\} \in \mathbb{C}$, $\{\phi_j, \psi_j\} \in \mathbb{R}^+$. An empty product, when it occurs, is taken to be one, so we get

$$n = 0 \longleftrightarrow \mathbb{W}(e) = 1, m = q \longleftrightarrow \mathbb{X}(e) = 1. n = p \longleftrightarrow \mathbb{Y}(e) = 1. \quad (19)$$

The $\mathcal{H}$ function is, in general, multivalued, but it can be made one-valued on the Riemann surface of $\log \mathfrak{g}$ by choosing a proper branch. We also note that when α and β are equal to 1, we obtain G functions $G^{m,n}_{\tau,\omega}(\mathfrak{g})$. The above integral representation of $\mathcal{H}$ functions, by involving products and ratios of Gamma functions, is known to be of the Mellin–Barnes integral type. A compact notation is usually adopted for (15).

$$\mathcal{H}^{m,n}_{\tau,\omega}(\mathfrak{g}) = \mathcal{H}^{m,n}_{\tau,\omega}\left[g \middle| \begin{matrix} \left(r_j, \alpha_j\right)_{j=1,\ldots,p} \\ \left(s_j, \beta_j\right)_{j=1,\ldots,p} \end{matrix}\right]. \quad (20)$$

Here, we assume $\mathfrak{F}_1 = [0, p]$, $\mathfrak{F}_2 = (0, +\infty)$, $\mathfrak{F}_3 = (0, 1]$, $\mathfrak{F}_4 = [0, +\infty]$, $\mathfrak{F}_5 = [0, 1]$ ($\mathfrak{F}_5{}^\circ = (0, 1)$), and $\mathfrak{F}_6 = [0, +\infty)$.

Assume that

$$\text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5) = \left\{ \begin{bmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{bmatrix} = \text{diag}[r_1, \ldots, r_n], r_1, \ldots, r_n \in \mathfrak{F}_5 \right\}, \quad (21)$$

in which

$$\mathbf{r} = \text{diag}[r_1, \ldots, r_n], \mathbf{s} = \text{diag}[s_1, \ldots, s_n] \in \text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5),$$
$$\mathbf{r} \le \mathbf{s} \Leftrightarrow r_i \le s_i \text{ for every } \quad i = 1, \ldots, n. \quad (22)$$

Also, $\mathbf{r} \prec \mathbf{s}$ denotes that $\mathbf{r} \prec \mathbf{s}$ and $\mathbf{r} \neq \mathbf{s}$; $\mathbf{r} \ll \mathbf{s}$ for every $i = 1, \ldots, n$. We define $\mathbf{a} = \text{diag}[a, \ldots, a]$ in $\text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5)$ where $a \in \mathfrak{F}_5$. Note that, $\text{diag}[1, \ldots, 1]$ is 1 and $\text{diag}[0, \ldots, 0]$ is 0.

Definition 6 (see [16–18]). A mapping $\circledast$: $\text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5) \times \text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5) \longrightarrow \text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5)$ is called a GTN if the following conditions are met:

(a) $(\forall \mathbf{r} \in (\text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5))(\mathbf{r} \circledast 1) = \mathbf{r})$ (boundary condition)
(b) $(\forall (\mathbf{r}, \mathbf{s}) \in (\text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5))^2)(\mathbf{r} \circledast \mathbf{s} = \mathbf{s} \circledast \mathbf{r})$ (commutativity)
(c) $(\forall (\mathbf{r}, \mathbf{s}, \mathbf{c}) \in (\text{diag}\mathfrak{M}_n(\mathfrak{F}_5))^3)(\mathbf{r} \circledast (\mathbf{s} \circledast \mathbf{c}) = (\mathbf{r} \circledast \mathbf{s}) \circledast \mathbf{c})$ (associativity)
(d) $(\forall (\mathbf{r}_1, \mathbf{s}_2, \mathbf{s}_1, \mathbf{s}_2) \in (\text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5))^4)$ $(\mathbf{r}_1 \prec \mathbf{r}_2$ and $\mathbf{s}_1 \prec \mathbf{s}_2$ implies that $\mathbf{r}_1 \circledast \mathbf{s}_1 \prec \mathbf{r}_2 \circledast \mathbf{s}_2)$ (monotonicity)

If for every $\mathbf{r}, \mathbf{s} \in \text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5)$ and each sequences $\{\mathbf{r}_k\}$ and $\{\mathbf{s}_k\}$ converging to $\mathbf{r}$ and $\mathbf{s}$, we get

$$\lim_k \left(\mathbf{r}_k \circledast \mathbf{s}_k\right) = \mathbf{r} \circledast \mathbf{r}, \quad (23)$$

and we conclude that the continuity of $\circledast$ on $\text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5)$ (CGTN).

(1) Define $\circledast_M$: $\text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5) \times \text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5) \longrightarrow \text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5)$, such that

$$\begin{aligned} \mathbf{r} \circledast_M \mathbf{s} &= \text{diag}[r_1, \ldots, r_n] \circledast_M \text{diag}[s_1, \ldots, s_n] \\ &= \text{diag}[\min\{r_1, s_1\}, \ldots, \min\{r_n, s_n\}], \end{aligned} \quad (24)$$

then $\circledast_M$ is CGTN (minimum CGTN).

(2) Define $\circledast_P$: $\text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5) \times \text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5) \longrightarrow \text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5)$, such that

$$\begin{aligned} \mathbf{r} \circledast_P \mathbf{s} &= \text{diag}[r_1, \ldots, r_n] \circledast_P \text{diag}[s_1, \ldots, s_n] \\ &= \text{diag}[r_1.s_1, \ldots, r_n.s_n], \end{aligned} \quad (25)$$

then $\circledast_P$ is CGTN (product CGTN).

(3) Define $\circledast_L$: $\text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5) \times \text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5) \longrightarrow \text{diag}\,\mathfrak{M}_n(\mathfrak{F}_5)$, such that

$$\begin{aligned} \mathbf{r} \circledast_L \mathbf{s} &= \text{diag}[r_1, \ldots, r_n] \circledast_L \text{diag}[s_1, \ldots, s_n] \\ &= \text{diag}[\max\{r_1 + s_1 - 1, 0\}, \ldots, \max\{r_n + s_n - 1, 0\}], \end{aligned} \quad (26)$$

then $\circledast_P$ is CGTN (Lukasiewicz CGTN).

Numerical examples of CGTN are as follows:

$$\operatorname{diag}\left[\frac{4}{7},\frac{7}{9},\frac{5}{9},\frac{3}{14},\frac{4}{3},\frac{6}{7}\right]\circledast_M\operatorname{diag}\left[\frac{3}{5},\frac{5}{18},\frac{6}{11},\frac{5}{7},\frac{13}{21},\frac{2}{3}\right]=\operatorname{diag}\left[\frac{4}{7},\frac{5}{18},\frac{6}{11},\frac{3}{14},\frac{13}{21},\frac{2}{3}\right]$$

or

$$\begin{bmatrix}\frac{4}{7}&&&&&\\&\frac{7}{9}&&&&\\&&\frac{5}{9}&&&\\&&&\frac{3}{14}&&\\&&&&\frac{4}{3}&\\&&&&&\frac{6}{7}\end{bmatrix}\circledast_M\begin{bmatrix}\frac{3}{5}&&&&&\\&\frac{5}{18}&&&&\\&&\frac{6}{11}&&&\\&&&\frac{5}{7}&&\\&&&&\frac{13}{21}&\\&&&&&\frac{2}{3}\end{bmatrix}=\begin{bmatrix}\frac{4}{7}&&&&&\\&\frac{5}{18}&&&&\\&&\frac{6}{11}&&&\\&&&\frac{3}{14}&&\\&&&&\frac{13}{21}&\\&&&&&\frac{2}{3}\end{bmatrix},$$

$$\operatorname{diag}\left[\frac{4}{7},\frac{7}{9},\frac{5}{9},\frac{3}{14},\frac{4}{3},\frac{6}{7}\right]\circledast_P\operatorname{diag}\left[\frac{3}{5},\frac{5}{18},\frac{6}{11},\frac{5}{7},\frac{13}{21},\frac{2}{3}\right]=\operatorname{diag}\left[\frac{12}{35},\frac{25}{62},\frac{10}{33},\frac{15}{98},\frac{52}{63},\frac{4}{7}\right]$$

or

$$\begin{bmatrix}\frac{4}{7}&&&&&\\&\frac{7}{9}&&&&\\&&\frac{5}{9}&&&\\&&&\frac{3}{14}&&\\&&&&\frac{4}{3}&\\&&&&&\frac{6}{7}\end{bmatrix}\circledast_P\begin{bmatrix}\frac{3}{5}&&&&&\\&\frac{5}{18}&&&&\\&&\frac{6}{11}&&&\\&&&\frac{5}{7}&&\\&&&&\frac{13}{21}&\\&&&&&\frac{2}{3}\end{bmatrix}=\begin{bmatrix}\frac{12}{35}&&&&&\\&\frac{35}{62}&&&&\\&&\frac{10}{33}&&&\\&&&\frac{15}{98}&&\\&&&&\frac{52}{63}&\\&&&&&\frac{4}{7}\end{bmatrix},$$

$$\operatorname{diag}\left[\frac{4}{7},\frac{7}{9},\frac{5}{9},\frac{3}{14},\frac{4}{3},\frac{6}{7}\right]\circledast_L\operatorname{diag}\left[\frac{3}{5},\frac{5}{18},\frac{6}{11},\frac{5}{7},\frac{13}{21},\frac{2}{3}\right]=\operatorname{diag}\left[\frac{6}{35},\frac{1}{18},\frac{10}{99},0,\frac{56}{63},\frac{11}{21}\right]$$

or

$$\begin{bmatrix} \frac{4}{7} & & & & & \\ & \frac{7}{9} & & & & \\ & & \frac{5}{9} & & & \\ & & & \frac{3}{14} & & \\ & & & & \frac{4}{3} & \\ & & & & & \frac{6}{7} \end{bmatrix} \circledast_L \begin{bmatrix} \frac{3}{5} & & & & & \\ & \frac{5}{18} & & & & \\ & & \frac{6}{11} & & & \\ & & & \frac{5}{7} & & \\ & & & & \frac{13}{21} & \\ & & & & & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{6}{35} & & & & & \\ & \frac{1}{18} & & & & \\ & & \frac{10}{99} & & & \\ & & & 0 & & \\ & & & & \frac{56}{63} & \\ & & & & & \frac{11}{21} \end{bmatrix}. \tag{27}$$

We get the following equations:

$$\begin{aligned} &\operatorname{diag}\left[\frac{4}{7},\frac{7}{9},\frac{5}{9},\frac{3}{14},\frac{4}{3},\frac{6}{7}\right] \circledast_M \operatorname{diag}\left[\frac{3}{5},\frac{5}{18},\frac{6}{11},\frac{5}{7},\frac{13}{21},\frac{2}{3}\right] \\ &\succcurlyeq \operatorname{diag}\left[\frac{4}{7},\frac{7}{9},\frac{5}{9},\frac{3}{14},\frac{4}{3},\frac{6}{7}\right] \circledast_P \operatorname{diag}\left[\frac{3}{5},\frac{5}{18},\frac{6}{11},\frac{5}{7},\frac{13}{21},\frac{2}{3}\right] \\ &\succcurlyeq \operatorname{diag}\left[\frac{4}{7},\frac{7}{9},\frac{5}{9},\frac{3}{14},\frac{4}{3},\frac{6}{7}\right] \circledast_L \operatorname{diag}\left[\frac{3}{5},\frac{5}{18},\frac{6}{11},\frac{5}{7},\frac{13}{21},\frac{2}{3}\right]. \end{aligned} \tag{28}$$

By considering the matrix-valued fuzzy function (MVFF) $\mathscr{G}$: $(\mathfrak{F}_1)^k \times \mathfrak{F}_2 \longrightarrow \operatorname{diag} M_n(\mathfrak{F}_3)$, then we have following conclusions:

(i) It is a left continuous and increasing function.

(ii) $\lim_{\mathfrak{y}\longrightarrow+\infty}\mathscr{G}(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k,\mathfrak{y})=1$ for any $(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\in\mathfrak{F}_1$ and $\mathfrak{y}\in\mathfrak{F}_2$.

(iii) For MVFFs $\mathscr{G}$ and $\mathscr{W}$, the relation "$\prec$" is defined as follows:

$$\begin{aligned} &\mathscr{G}\precsim\mathscr{W}\Leftrightarrow\mathfrak{G}(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k,\mathfrak{y}) \\ &\quad\leq\mathscr{W}(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k,\mathfrak{y}), \\ &\forall\mathfrak{y}\in\mathfrak{F}_2 \text{ and } (\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\in\mathfrak{F}_1. \end{aligned} \tag{29}$$

Definition 7. Let $\circledast$ be a CGTN, $\mathscr{J}$ be a vector space, and $\mathscr{N}$: $\mathscr{J}^k\times\mathfrak{F}_2\longrightarrow\operatorname{diag}M_n(\mathfrak{F}_3)$ be a matrix-valued fuzzy set (MVFS). Triple $(\mathscr{J},\mathfrak{Y},\circledast)$ is called a matrix-valued fuzzy k-normed space (MVFkN-space) if

(MVFkN1) $\mathscr{N}(\mathfrak{c}_1-\mathfrak{c}_1',\ldots,\mathfrak{c}_k-\mathfrak{c}_{kk}',\mathfrak{y})=1$ if and only if $(\mathfrak{c}_1-\mathfrak{c}_1',\ldots,\mathfrak{c}_k-\mathfrak{c}_k')$ are linearly dependent and $\mathfrak{y}\in\mathfrak{F}_2$;

(MVFkN2) $\mathscr{N}(\varkappa(\mathfrak{c}_1-\mathfrak{c}_1',\ldots,\mathfrak{c}_k-\mathfrak{c}_k'),\mathfrak{y})=\mathscr{N}(\mathfrak{c}_1-\mathfrak{c}_1',\ldots,\mathfrak{c}_k-\mathfrak{c}_k',\mathfrak{y}/|\varkappa|)$ for all $(\mathfrak{c}_1-\mathfrak{c}_1',\ldots,\mathfrak{c}_k-\mathfrak{c}_k')\in\mathscr{J}$ and $\varkappa\in\mathbb{C}$ with $\varkappa\neq0$;

(MVFkN3) - $\mathscr{N}(\mathfrak{c}_0+\mathfrak{c}_1-\mathfrak{c}_1',\ldots,\mathfrak{c}_k-\mathfrak{c}_k',\mathfrak{y}+\mathfrak{z})\succcurlyeq\mathscr{N}(\mathfrak{c}_0,\mathfrak{c}_2-\mathfrak{c}_2',\ldots,\mathfrak{c}_k-\mathfrak{c}_k',\mathscr{N})\circledast\mathscr{N}(\mathfrak{c}_1-\mathfrak{c}_1',\mathfrak{c}_2-\mathfrak{c}_2',\ldots,\mathfrak{c}_k-\mathfrak{c}_k',\mathfrak{z})$ for all $(\mathfrak{c}_1-\mathfrak{c}_1',\ldots,\mathfrak{c}_k-\mathfrak{c}_k')\in\mathfrak{F}$ and any $\mathfrak{y},\mathfrak{z}\in\mathfrak{F}_2$;

(MVFkN4) $\lim_{\mathfrak{y}\longrightarrow+\infty}\mathscr{N}(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k,\mathfrak{y})=1$ for any $\mathfrak{y}\in\mathfrak{F}_2$.

When an MVFkN-space is complete, we denote it by an MVFkB-space. Using the concept of the H-Fox function, we define an MVF H-Fox function $\mathbf{H}_{\tau,\omega}^{\mathbf{m,n}}$: $(\mathfrak{F}_1)^k\times\mathfrak{F}_2\longrightarrow\operatorname{diag}M_n(\mathfrak{F}_3)(\tau,\omega\in\mathfrak{F}_2)$ as a control function in the MVFkN-spaces as follows:

$$\begin{aligned} &\mathbf{H}_{\tau,\omega}^{\mathbf{m,n}}\left(-\frac{|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|}{\mathfrak{y}}\right) \\ &= \operatorname{diag}\left[H_{\tau,\omega}^{m,n}\left[\left(-\frac{|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|}{\mathfrak{y}}\right)\middle|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right], H_{\tau,\omega}^{m,n}\left[\left(-\frac{|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|}{\mathfrak{y}}\right)\middle|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right]\right. \\ &\quad\left.,\ldots,H_{\tau,\omega}^{m,n}\left[\left(-\frac{|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|}{\mathfrak{y}}\right)\middle|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right]\right]. \end{aligned} \tag{30}$$

For an MVF H-Fox function $\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}$, we have

(1) It is a left continuous and increasing function for positive values.

(2) $\lim_{\mathfrak{y}\longrightarrow+\infty}\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}(-|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|/\mathfrak{y})=1$.

(3) For $\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}$ and also, for the matrix-valued fuzzy function $\Phi_{\tau,\omega}$, we have

$$\begin{aligned}\mathbf{\Phi}_{\tau,\omega}\precsim\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}\Leftrightarrow\mathbf{\Phi}_{\tau,\omega}\left(-\frac{|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|}{\mathfrak{y}}\right)\\ \le\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}\left(-\frac{|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|}{\mathfrak{y}}\right),\end{aligned}\tag{31}$$

and also, we have

(1) $\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}(-|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|/\mathfrak{y})\succ 0$.

(2) We can easily show that for $\mathfrak{y}\in\mathfrak{F}_2$, $\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}(-|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|/\mathfrak{y})=1\Leftrightarrow\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k=0$.

(3) We show that

$$\begin{aligned}\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}\left(-\frac{|\alpha(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)|}{\mathfrak{y}}\right)\\ =\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}\left(-\frac{|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)|}{\mathfrak{y}/|\alpha|}\right).\end{aligned}\tag{32}$$

Then, we have

$$\begin{aligned}&\mathbf{H}^{\mathbf{m,n}}_{\boldsymbol{\tau},\boldsymbol{\omega}}\left(-\frac{|\alpha(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)|}{\mathfrak{y}}\right)\\
&=\operatorname{diag}\left[H^{m,n}_{\tau,\omega}\left[\left(-\frac{|\alpha(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)|}{\mathfrak{y}}\right)\middle|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right],H^{m,n}_{\tau,\omega}\left[\left(-\frac{|\alpha(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)|}{\mathfrak{y}}\right)\middle|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right]\right.\\
&\left.,\ldots,H^{m,n}_{\tau,\omega}\left[\left(-\frac{|\alpha(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)|}{\mathfrak{y}}\right)\middle|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right]\right]\\
&=\operatorname{diag}\left[H^{m,n}_{\tau,\omega}\left[\left(-\frac{|\alpha(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)|}{\mathfrak{y}}\right)\middle|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right],H^{m,n}_{\tau,\omega}\left[\left(-\frac{|\alpha(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)|}{\mathfrak{y}}\right)\middle|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right]\right.\\
&\left.,\ldots,H^{m,n}_{\tau,\omega}\left[\left(-\frac{|\alpha(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)|}{\mathfrak{y}}\right)\middle|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right]\right]\\
&=\operatorname{diag}\left[H^{m,n}_{\tau,\omega}\left[\left(\frac{-|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)|}{\mathfrak{y}/|\alpha|}\right)\middle|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right],H^{m,n}_{\tau,\omega}\left[\left(\frac{-|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)|}{\mathfrak{y}/|\alpha|}\right)\middle|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right]\right.\\
&\left.,\ldots,H^{m,n}_{\tau,\omega}\left[\left(\frac{-|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)|}{\mathfrak{y}/|\alpha|}\right)\middle|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right]\right]\\
&=\mathbf{H}^{\mathbf{m,n}}_{\boldsymbol{\tau},\boldsymbol{\omega}}\left(-\frac{|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)|}{\mathfrak{y}/|\alpha|}\right).\end{aligned}\tag{33}$$

(4) We show that

$$\mathbf{H}_{\tau,\omega}^{\mathbf{m,n}}\left(-\frac{\left|((\varpi_1-\vartheta_1)+(\mathfrak{c}z_1-\mathfrak{c}p_1),\ldots,(\varpi_k-\vartheta_k)+(\mathfrak{c}z_k-\mathfrak{c}p_k))\right|}{\mathfrak{y}+\mathfrak{s}}\right)$$
$$\succcurlyeq \mathbf{H}_{\tau,\omega}^{\mathbf{m,n}}\left(-\frac{\left|((\varpi_1-\vartheta_1),\ldots,(\varpi_k-\vartheta_k))\right|}{\mathfrak{y}}\right)$$
$$\circledast \mathbf{H}_{\tau,\omega}^{\mathbf{m,n}}\left(-\frac{\left|(\mathfrak{c}z_1-\mathfrak{c}p_1),\ldots,(\mathfrak{c}z_k-\mathfrak{c}p_k)\right|}{\mathfrak{s}}\right). \tag{34}$$

Suppose that $H_{\tau,\omega}^{m,n}\left[-(|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|/\mathfrak{y})\left|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right.\right] \leq H_{\tau,\omega}^{m,n}\left[-(|\mathfrak{c}z_1-\mathfrak{c}p_1,\ldots,\mathfrak{c}z_k-\mathfrak{c}p_k|/\mathfrak{s})\left|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right.\right]$, then we have

$$\begin{gathered}
\frac{-\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\right|}{y}\leq\frac{-\left|(\mathfrak{c}z_1-\mathfrak{c}p_1,\ldots,\mathfrak{c}z_k-\mathfrak{c}p_k)\right|}{s}\Rightarrow\\
\frac{\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\right|}{y}\geq\frac{\left|(\mathfrak{c}z_1-\mathfrak{c}p_1,\ldots,\mathfrak{c}z_k-\mathfrak{c}p_k)\right|}{s}\Rightarrow\\
\frac{\mathfrak{s}\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\right|}{y}\geq\left|(\mathfrak{c}z_1-\mathfrak{c}p_1,\ldots,\mathfrak{c}z_k-\mathfrak{c}p_k)\right|\Rightarrow\\
\frac{\mathfrak{s}\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\right|}{y}+\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\right|\geq\\
\left|(\mathfrak{c}z_1-\mathfrak{c}p_1,\ldots,\mathfrak{c}z_k-\mathfrak{c}p_k)\right|+\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\right|\geq\\
\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)+(\mathfrak{c}z_1-\mathfrak{c}p_1,\ldots,\mathfrak{c}z_k-\mathfrak{c}p_k)\right|\Rightarrow\\
\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\right|\left(\frac{s}{y}+1\right)\geq\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)+(\mathfrak{c}z_1-\mathfrak{c}p_1,\ldots,\mathfrak{c}z_k-\mathfrak{c}p_k)\right|\Rightarrow\\
\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\right|\left(\frac{s+y}{y}\right)\geq\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)+(\mathfrak{c}z_1-\mathfrak{c}p_1,\ldots,\mathfrak{c}z_k-\mathfrak{c}p_k)\right|\Rightarrow\\
\frac{\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\right|}{y}\geq\frac{\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)+(\mathfrak{c}z_1-\mathfrak{c}p_1,\ldots,\mathfrak{c}z_k-\mathfrak{c}p_k)\right|}{s+y}\Rightarrow\\
\frac{-\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\right|}{y}\leq\frac{-\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)+(\mathfrak{c}z_1-\mathfrak{c}p_1,\ldots,\mathfrak{c}z_k-\mathfrak{c}p_k)\right|}{s+y}\Rightarrow\\
H_{\tau,\omega}^{m,n}\left[\frac{-\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)+(\mathfrak{c}z_1-\mathfrak{c}p_1,\ldots,\mathfrak{c}z_k-\mathfrak{c}p_k)\right|}{s+y}\left|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right.\right]\\
\geq H_{\tau,\omega}^{m,n}\left[\frac{-\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\right|}{y}\left|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right.\right].
\end{gathered} \tag{35}$$

Therefore, we have

$$\begin{aligned}
&\operatorname{diag}\left[H^{m,n}_{\tau,\omega}\left[\frac{\dashv\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)+(\mathfrak{s}_1-\mathfrak{p}_1,\ldots,\mathfrak{s}_k-\mathfrak{p}_k)\right|}{\mathfrak{s}+\mathfrak{y}}\Bigg|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\ \\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right],\right.\\
&H^{m,n}_{\tau,\omega}\left[\frac{\dashv\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)+(\mathfrak{s}_1-\mathfrak{p}_1,\ldots,\mathfrak{s}_k-\mathfrak{p}_k)\right|}{\mathfrak{s}+\mathfrak{y}}\Bigg|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\ \\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right],\ldots,\\
&\left.H^{m,n}_{\tau,\omega}\left[\frac{\dashv\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)+(\mathfrak{s}_1-\mathfrak{p}_1,\ldots,\mathfrak{s}_k-\mathfrak{p}_k)\right|}{\mathfrak{s}+\mathfrak{y}}\Bigg|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\ \\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right]\right]\succcurlyeq\\
&\operatorname{diag}\left[H^{m,n}_{\tau,\omega}\left[\frac{\dashv\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\right|}{\mathfrak{y}}\Bigg|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\ \\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right],H^{m,n}_{\tau,\omega}\left[\frac{\dashv\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\right|}{\mathfrak{y}}\Bigg|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\ \\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right]\right.\\
&\left.,\ldots,H^{m,n}_{\tau,\omega}\left[\frac{\dashv\left|(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\right|}{\mathfrak{y}}\Bigg|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\ \\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right]\right]\circledast\\
&\operatorname{diag}\left[H^{m,n}_{\tau,\omega}\left[\frac{\dashv\left|\mathfrak{s}_1-\mathfrak{p}_1,\ldots,\mathfrak{s}_k-\mathfrak{p}_k\right|}{\mathfrak{s}}\Bigg|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\ \\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right],\right.\\
&H^{m,n}_{\tau,\omega}\left[\frac{\dashv\left|\mathfrak{s}_1-\mathfrak{p}_1,\ldots,\mathfrak{s}_k-\mathfrak{p}_k\right|}{\mathfrak{s}}\Bigg|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\ \\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right]\\
&\left.,\ldots,H^{m,n}_{\tau,\omega}\left[\frac{\dashv\left|\mathfrak{s}_1-\mathfrak{p}_1,\ldots,\mathfrak{s}_k-\mathfrak{p}_k\right|}{\mathfrak{s}}\Bigg|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\ \\(b_j,\beta_j)_{j=1,\ldots,p}\end{matrix}\right]\right].
\end{aligned}\tag{36}$$

Consequently, if

$$\mathfrak{Y}(\mathfrak{z}_1-\mathfrak{z}_1',\ldots,\mathfrak{z}_k-\mathfrak{z}_k',\mathfrak{y})=\operatorname{diag}\begin{bmatrix}H^{m,n}_{\tau,\omega}\left(-\dfrac{\left|\mathfrak{z}_1-\mathfrak{z}_1',\ldots,\mathfrak{z}_k-\mathfrak{z}_k'\right|}{\mathfrak{y}}\right),\\ H^{m,n}_{\tau,\omega}\left(-\dfrac{\left|\mathfrak{z}_1-\mathfrak{z}_1',\ldots,\mathfrak{z}_k-\mathfrak{z}_k'\right|}{\mathfrak{y}}\right),\\ \ldots,H^{m,n}_{\tau,\omega}\left(-\dfrac{\left|\mathfrak{z}_1-\mathfrak{z}_1',\ldots,\mathfrak{z}_k-\mathfrak{z}_k'\right|}{\mathfrak{y}}\right)\end{bmatrix},\tag{37}$$

for $\mathfrak{y}\in\mathfrak{F}_2$, then $(\mathscr{J},\mathfrak{Y},\circledast_M)$ is an MVFkN-space. From now on, we assume $\circledast=\circledast_M$.

Theorem 3 (see [14, 19]). *We consider the $\mathfrak{F}_4$-valued metric space $(\mathscr{A},d)$. For $\mathfrak{f},\mathfrak{h}\in\mathscr{A}$, we consider the self-map Π on $\mathscr{A}$ such that*

$$d(\Pi\mathfrak{f},\Xi\mathfrak{h})\le\nu\, d(\mathfrak{h},\mathfrak{f}),\tag{38}$$

where $\nu<1$ is a Lipschitz constant. Let $\mathfrak{f}\in\mathscr{A}$. Therefore, we have following two ways:

(i) $d(\Pi^{\mathbf{e}}\mathfrak{f},\Pi^{\mathbf{e}+1}\mathfrak{f})=\infty,\quad\forall\mathbf{e}\in\mathbb{N}$

or

(ii) we can find $\mathbf{e}_0\in\mathbb{N}$ such that: $d(\Pi^{\mathbf{e}}\mathfrak{f},\Pi^{\mathbf{e}+1}\mathfrak{f})<\infty,\quad\forall\mathbf{e}\ge\mathbf{e}_0$

If condition (ii) is true for us, then we have following conclusions:

(1) *The fixed point* $\mathfrak{h}^*$ *of* Π *is the convergence point of the sequence* $\{\Pi^{e}\mathfrak{f}\}$

(2) *In the set* $\mathscr{H}^* = \{\mathfrak{h} \in \mathscr{A} \mid d(\Pi^{e_0}\mathfrak{f}, \mathfrak{h}) < \infty\}$, $\mathfrak{h}^*$ *is the unique fixed point of* Π

(3) $(1-\nu)d(\mathfrak{h},\mathfrak{h}^*) \leq d(\mathfrak{h},\Pi\mathfrak{h})$ *for every* $\mathfrak{h} \in \mathfrak{A}$

Definition 8. Let function $\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}$ be an MVF function. Equation (1) is said to be Hyers–Ulam-H-Fox stable, and if $\mathfrak{h}(\varpi_1-\vartheta_1),\ldots,\mathfrak{h}(\varpi_k-\vartheta_k)$ is a given differentiable function, we obtain the following equation:

$$\begin{aligned}&\mathscr{N}(\mathfrak{D}^{a}_{\omega}\mathfrak{h}(\varpi_1-\vartheta_1)-\varrho\mathfrak{h}(\varpi_1-\vartheta_1)-K(\varpi_1-\vartheta_1,\mathfrak{h}(\varpi_1-\vartheta_1)),\ldots,\\&\mathfrak{D}^{a}_{\omega}\mathfrak{h}(\varpi_k-\vartheta_k)-\varrho\mathfrak{h}(\varpi_k-\vartheta_k)-K(\varpi_k-\vartheta_k,\mathfrak{h}(\varpi_k-\vartheta_k)),\aleph)\\&\succcurlyeq\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}\left(-\frac{|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|}{\aleph}\right),\end{aligned} \tag{39}$$

for $(\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k)\in\mathfrak{F}_1$, and we can find a solution $\mathfrak{f}(\varpi_1-\vartheta_1)\ldots,\mathfrak{f}(\varpi_k-\vartheta_k)$ of (1) such that for some $\eta>0$, which is as follows:

$$\begin{aligned}&\mathscr{N}(\mathfrak{h}(\varpi_1-\vartheta_1)-\mathfrak{f}(\varpi_1-\vartheta_1),\ldots,\mathfrak{h}(\varpi_k-\vartheta_k)\\&\quad-\mathfrak{f}(\varpi_k-\vartheta_k),\aleph)\succcurlyeq\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}\left(-\frac{|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|}{\aleph/\eta}\right).\end{aligned} \tag{40}$$

Remark 2. Let $\mathfrak{h}$ be a solution of inequality (39). Then, $\mathfrak{h}$ is a solution of the following integral inequality:

$$\begin{aligned}&\mathscr{N}\Bigg(\mathfrak{h}(\varpi_1-\vartheta_1)-\mathfrak{h}_a\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_1-\vartheta_1)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\\&-\frac{1}{\Gamma(\omega)}\int_a^{\varpi-\vartheta}\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_1-\vartheta_1)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\sum_{\varsigma=0}^{\infty}\frac{(((\lambda_1-\lambda_1')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}K((\lambda_1-\lambda_1'),\mathfrak{h}(\varpi_1-\vartheta_1))\\&\cdot\left(\frac{(\lambda_1-\lambda_1')}{t}\right)^{s}\left(\log\frac{t}{(\lambda_1-\lambda_1')}\right)^{\omega-1}\frac{d(\lambda_1-\lambda_1')}{(\lambda_1-\lambda_1')},\ldots,\mathfrak{h}(\varpi_k-\vartheta_k)-\mathfrak{h}_a\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\\&-\frac{1}{\Gamma(\omega)}\int_a^{\varpi-\vartheta}\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\sum_{\varsigma=0}^{\infty}\frac{(((\lambda_k-\lambda_k')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}K((\lambda_k-\lambda_k'),\mathfrak{h}(\varpi_k-\vartheta_k))\\&\cdot\left(\frac{(\lambda_k-\lambda_k')}{t}\right)^{s}\left(\log\frac{t}{(\lambda_k-\lambda_k')}\right)^{\omega-1}\frac{d(\lambda_k-\lambda_k')}{(\lambda_k-\lambda_k')},\aleph\Bigg)\\&\succcurlyeq\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}\Bigg(|\frac{1}{\Gamma(\omega)}\int_a^{\varpi-\vartheta}\sum_{\varsigma=0}^{\infty}\frac{((\lambda_1-\lambda_1'-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\\&\left(\frac{(\lambda_1-\lambda_1')}{t}\right)^{s}\left(\log\frac{t}{(\lambda_1-\lambda_1')}\right)^{\omega-1}\frac{d(\lambda_1-\lambda_1')}{(\lambda_1-\lambda_1')},\ldots,\frac{1}{\Gamma(\omega)}\int_a^{\varpi-\vartheta}\sum_{\varsigma=0}^{\infty}\frac{(((\lambda_k-\lambda_k')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\\&\cdot\left(\frac{(\lambda_k-\lambda_k')}{t}\right)^{s}\left(\log\frac{t}{(\lambda_k-\lambda_k')}\right)^{\omega-1}\frac{d(\lambda_k-\lambda_k')}{(\lambda_k-\lambda_k')}|\frac{1}{\aleph/\digamma\sum_{\varsigma=0}^{\infty}((b-a)^{\omega}/\omega)^{\varsigma\varrho}/\Gamma(1+\tau\varsigma\varrho)}\Bigg),\end{aligned} \tag{41}$$

for every $\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k\in\mathfrak{F}_1$ and $\aleph\in\mathfrak{F}_2$.

3. Hyers–Ulam-H-Fox Stability for the Conformable Fractional Differential Equation

Now, we use the fixed-point method based on Theorem 3 to show that equation (1) is Hyers–Ulam-H-Fox stable [14] in the MVFkB-space $(\mathscr{J}, \mathscr{N}, \circledast)$ with MVFF $\mathbf{H}^{\mathbf{m},\mathbf{n}}_{\tau,\omega}$ [20–22].

We set the set $\mathscr{H}$ as follows:

$$\mathscr{H} = \{\mathfrak{f}\colon \mathfrak{F}_1 \longrightarrow \mathscr{J}, \mathfrak{f} \text{ is differentiable}\}, \tag{42}$$

and we consider the mapping $d\colon \mathscr{H} \times \mathscr{H} \longrightarrow \mathfrak{F}_4$ as

$$\begin{aligned} d(\mathfrak{f}, \mathfrak{h}) = \inf\{&\mathfrak{R} \in \mathfrak{F}_6\colon \mathscr{N}(\mathfrak{h}(\varpi_1 - \vartheta_1) - \mathfrak{f}(\varpi_1 - \vartheta_1), \ldots, \mathfrak{h}(\varpi_k - \vartheta_k) - \mathfrak{f}(\varpi_k - \vartheta_k), \aleph) \\ &\succcurlyeq \mathbf{H}^{\mathbf{m},\mathbf{n}}_{\tau,\omega}\left(-\frac{|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k|}{\aleph/\mathfrak{R}}\right), \\ &\forall \mathfrak{h}, \mathfrak{f} \in \mathscr{H}, (\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k) \in \mathfrak{F}_1, \aleph \in \mathfrak{F}_2\}. \end{aligned} \tag{43}$$

Theorem 4. *$(\mathscr{H}, d)$ is a complete $\mathfrak{F}_4$-valued metric space.*

Proof. We have $d(\mathfrak{h}, \mathfrak{f}) = 0$ if and only if $\mathfrak{h} = \mathfrak{f}$. Assume that $d(\mathfrak{h}, \mathfrak{f}) = 0$, then we have

$$\begin{aligned} \inf\{&\mathfrak{R} \in \mathfrak{F}_6\colon \mathscr{N}(\mathfrak{h}(\varpi_1 - \vartheta_1) - \mathfrak{f}(\varpi_1 - \vartheta_1), \ldots, \mathfrak{h}(\varpi_k - \vartheta_k) - \mathfrak{f}(\varpi_k - \vartheta_k), \aleph) \\ &\succcurlyeq \mathbf{H}^{\mathbf{m},\mathbf{n}}_{\tau,\omega}\left(-\frac{|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k|}{\aleph/\mathfrak{R}}\right), \\ &\forall \mathfrak{h}, \mathfrak{f} \in \mathscr{H}, (\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k) \in \mathfrak{F}_1, \aleph \in \mathfrak{F}_2\} = 0, \end{aligned} \tag{44}$$

so

$$\begin{aligned} &\mathscr{N}(\mathfrak{h}(\varpi_1 - \vartheta_1) - \mathfrak{f}(\varpi_1 - \vartheta_1), \ldots, \mathfrak{h}(\varpi_k - \vartheta_k) - \mathfrak{f}(\varpi_k - \vartheta_k), \aleph) \\ &\succcurlyeq \mathbf{H}^{\mathbf{m},\mathbf{n}}_{\tau,\omega}\left(-\frac{|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k|}{\aleph/\mathfrak{R}}\right), \end{aligned} \tag{45}$$

for all $\mathfrak{R} \in \mathfrak{F}_6$. We assume $\mathfrak{R}$ to zero in the above inequality, and we get

$$\mathscr{N}(\mathfrak{h}(\varpi_1 - \vartheta_1) - \mathfrak{f}(\varpi_1 - \vartheta_1), \ldots, \mathfrak{h}(\varpi_k - \vartheta_k) - \mathfrak{f}(\varpi_k - \vartheta_k), \aleph) = 1. \tag{46}$$

Thus, $\mathfrak{h}(\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k) = \mathfrak{f}(\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k)$ for every $(\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k) \in \mathfrak{F}_1$ and vice versa. Also, we have $d(\mathfrak{h}, \mathfrak{f}) = d(\mathfrak{f}, \mathfrak{h})$ for every $\mathfrak{h}, \mathfrak{f} \in \mathscr{H}$. Now, let $d(\mathfrak{h}, \mathfrak{f}) = \alpha_1 \in \mathfrak{F}_2$ and $d(\mathfrak{h}, \mathfrak{f}) = \alpha_2 \in \mathfrak{F}_2$. Then, we have

$$\begin{aligned} &\mathscr{N}(\mathfrak{h}(\varpi_1 - \vartheta_1) - \mathfrak{f}(\varpi_1 - \vartheta_1), \ldots, \mathfrak{h}(\varpi_k - \vartheta_k) - \mathfrak{f}(\varpi_k - \vartheta_k), \aleph) \succcurlyeq \mathbf{H}^{\mathbf{m},\mathbf{n}}_{\tau,\omega}\left(-\frac{|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k|}{\aleph/\alpha_1}\right), \\ &\mathscr{N}(\mathfrak{h}(\varpi_1 - \vartheta_1) - \mathfrak{f}(\varpi_1 - \vartheta_1), \ldots, \mathfrak{h}(\varpi_k - \vartheta_k) - \mathfrak{f}(\varpi_k - \vartheta_k), \aleph) \succcurlyeq \mathbf{H}^{\mathbf{m},\mathbf{n}}_{\tau,\omega}\left(-\frac{|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k|}{\aleph/\alpha_2}\right), \end{aligned} \tag{47}$$

for every $\aleph \in \mathfrak{F}_2$. Then, we have

$$\begin{aligned} &\mathscr{N}(\mathfrak{h}(\varpi_1 - \vartheta_1) - \mathfrak{p}(\varpi_1 - \vartheta_1), \ldots, \mathfrak{h}(\varpi_k - \vartheta_k) - \mathfrak{p}(\varpi_k - \vartheta_k), (\alpha_1 + \alpha_2)\aleph) \\ &\succcurlyeq [\mathscr{N}(\mathfrak{h}(\varpi_1 - \vartheta_1) - \mathfrak{f}(\varpi_1 - \vartheta_1), \ldots, \mathfrak{h}(\varpi_k - \vartheta_k) - \mathfrak{f}(\varpi_k - \vartheta_k), (\alpha_1)\aleph) \\ &\circledast \mathscr{N}(\mathfrak{f}(\varpi_1 - \vartheta_1) - \mathfrak{p}(\varpi_1 - \vartheta_1), \ldots, \mathfrak{f}(\varpi_k - \vartheta_k) - \mathfrak{p}(\varpi_k - \vartheta_k), (\alpha_2)\aleph)] \\ &\succcurlyeq \mathbf{H}^{\mathbf{m},\mathbf{n}}_{\tau,\omega}\left(-\frac{|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k|}{\aleph}\right) \circledast \mathbf{H}^{\mathbf{m},\mathbf{n}}_{\tau,\omega}\left(-\frac{|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k|}{\aleph}\right) \\ &= \mathbf{H}^{\mathbf{m},\mathbf{n}}_{\tau,\omega}\left(-\frac{|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k|}{\aleph}\right), \end{aligned} \tag{48}$$

so $d(\mathfrak{h}, (\mathfrak{p})) \le \alpha_1 + \alpha_2$. Thus, $d(\mathfrak{h}, (\mathfrak{p})) \le d(\mathfrak{h}, (\mathfrak{f})) + d(\mathfrak{f}, (\mathfrak{p}))$. To show the completeness of $(\mathscr{H}, d)$, we suppose that $\{\mathfrak{h}_k\}_k$ is a Cauchy sequence in $(\mathscr{H}, d)$. Let $\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k \in \mathfrak{F}_1$. Assume that $\gamma \in \mathfrak{F}_2$ and $\theta \in \mathfrak{F}_5{}^\circ$ are arbitrary and consider $\aleph \in \mathfrak{F}_2$ such that $\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}(-|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k|/\aleph) \succ 1 - \theta$. For $\alpha\aleph < \gamma$, we choose $k_0 \in \mathbb{N}$ such that

$$d(\mathfrak{h}_k, \mathfrak{h}_\ell) < \alpha \ \forall k, \ell \ge k_0. \tag{49}$$

Then, we have

$$\begin{aligned}
&\mathscr{N}(\mathfrak{h}_k(\varpi_1 - \vartheta_1) - \mathfrak{h}_\ell(\varpi_1 - \vartheta_1), \ldots, \mathfrak{h}_k(\varpi_k - \vartheta_k) - \mathfrak{h}_\ell(x_k - y_k), \gamma)\\
&\succcurlyeq \mathscr{N}(\mathfrak{h}_k(\varpi_1 - \vartheta_1) - \mathfrak{h}_\ell(\varpi_1 - \vartheta_1), \ldots, \mathfrak{h}_k(\varpi_k - \vartheta_k) - \mathfrak{h}_\ell(\varpi_k - \vartheta_k), \alpha\aleph)\\
&\succcurlyeq \mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}\left(-\frac{|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k|}{\aleph}\right)\\
&\succ 1 - \boldsymbol{\theta}.
\end{aligned} \tag{50}$$

So we have the following equation:

$$\mathscr{N}(\mathfrak{h}(\varpi_1 - \vartheta_1) - \mathfrak{h}(\varpi_1 - \vartheta_1), \ldots, \mathfrak{h}(\varpi_k - \vartheta_k) - \mathfrak{h}(\varpi_k - \vartheta_k), \gamma) \succ 1 - \boldsymbol{\theta}, \tag{51}$$

which implies that the sequence $\{\mathfrak{h}_k(\varpi_1 - \vartheta_1), \mathfrak{h}_k(\varpi_2 - \vartheta_2), \ldots, \mathfrak{h}_k(\varpi_k - \vartheta_k)\}_k$ is Cauchy in a complete space $(\mathscr{J}, \mathscr{N}, \circledast)$ on a compact set $\mathfrak{F}_1$. Then, it is uniformly convergent to the mapping $\mathfrak{h}: \mathfrak{F}_1 \longrightarrow \mathscr{J}$. By uniform convergence property, we conclude that $\mathfrak{h}$ is differentiable, i.e., an element of $\mathscr{H}$, and then, $(\mathscr{H}, d)$ is complete. □

Now, we can investigate Hyers–Ulam-H-Fox stability and get an approximation for the solution of conformable FDE (1). In [23–48], there are new stability problems that one can prove them by our method. □

Theorem 5. *Let $(\mathscr{J}, \mathscr{N}, \circledast)$ be an MVFkB-space and consider the constant coefficient* ϱ, *E, and* $\wp$. *Then, we have* $\beth = \wp\left(\sum_{\varsigma=0}^{\infty} \sum_{\varsigma_1 \cdots \varsigma_k = \varsigma}^{\infty} \binom{\varsigma}{\varsigma_1, \ldots, \varsigma_k} (((\varpi_1 - \vartheta_1) - a)^\omega/\omega)^{\varsigma_1\varrho} \cdots (((\varpi_k - \vartheta_k) - a)^\omega/\omega)^{\varsigma_k\varrho}/\Gamma(\varsigma_1 \cdots \varsigma_k + \tau\varsigma\varrho)\right)E < 1.$

Suppose that the following conditions hold:

(1) For continuous function $K: \mathfrak{F}_1 \times \mathscr{J} \longrightarrow \mathscr{J}$, we obtain

$$\begin{aligned}
&\mathscr{N}(K(\varpi_1 - \vartheta_1, \mathfrak{f}(\varpi_1 - \vartheta_1)) - K(\varpi_1 - \vartheta_1, \mathfrak{h}(\varpi_1 - \vartheta_1)),\\
&\ldots, K(\varpi_k - \vartheta_k, \mathfrak{f}(\varpi_k - \vartheta_k)) - K(\varpi_k - \vartheta_k, \mathfrak{h}(\varpi_k - \vartheta_k)), \aleph)\\
&\succcurlyeq \mathscr{N}\left(\mathfrak{h}(\varpi_1 - \vartheta_1) - \mathfrak{f}(\varpi_1 - \vartheta_1), \ldots, \mathfrak{h}(\varpi_k - \vartheta_k) - \mathfrak{f}(\varpi_1 - \vartheta_1), \frac{\aleph}{\mathfrak{R}}\right),
\end{aligned} \tag{52}$$

(2) MVFF $\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}$ satisfying the following equation:

$$\begin{aligned}
&\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}\Bigg(\Big|\frac{1}{\Gamma(\omega)}\Bigg\{\int_a^{(\varpi_1-\vartheta_1)} \sum_{\varsigma=0}^{\infty} \frac{\left(((\lambda_1 - \lambda_1') - a)^\omega/\omega\right)^{-\varsigma\varrho}}{\Gamma(1 + \tau\varsigma\varrho)} \left(\frac{(\lambda_1 - \lambda_1')}{t}\right)^s \left(\log\frac{t}{(\lambda_1 - \lambda_1')}\right)^{\omega-1} \frac{d(\lambda_1 - \lambda_1')}{(\lambda_1 - \lambda_1')},\\
&\ldots, \int_a^{(\varpi_k-\vartheta_k)} \sum_{\varsigma=0}^{\infty} \frac{\left(((\lambda_k - \lambda_k') - a)^\omega/\omega\right)^{-\varsigma\varrho}}{\Gamma(1 + \tau\varsigma\varrho)} \left(\frac{(\lambda_k - \lambda_k')}{t}\right)^s \left(\log\frac{t}{(\lambda_k - \lambda_k')}\right)^{\omega-1} \frac{d(\lambda_k - \lambda_k')}{(\lambda_k - \lambda_k')}\Bigg]\Big|\frac{1}{\aleph}\Bigg)\\
&\succcurlyeq \mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}\left(-\frac{|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k|}{\aleph/E}\right).
\end{aligned} \tag{53}$$

for every $\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k \in \mathfrak{F}_1$ and $\aleph \in \mathfrak{F}_2$.

Let $\mathfrak{h}: \mathfrak{F}_1 \longrightarrow \mathscr{J}$ be a differentiable function satisfying the following equation:

$$\begin{aligned}
&\mathscr{N}(\mathfrak{D}^{a}_{\omega}\mathfrak{h}(\varpi_1 - \vartheta_1) - \varrho\mathfrak{h}(\varpi_1 - \vartheta_1) - \mathfrak{k}(\varpi_1 - \vartheta_1, \mathfrak{h}(\varpi_1 - \vartheta_1)),\\
&\ldots, \mathfrak{D}^{a}_{\omega}\mathfrak{h}(\varpi_k - \vartheta_k) - \lambda\mathfrak{h}(\varpi_k - \vartheta_k) - K(\varpi_k - \vartheta_k, \mathfrak{h}(\varpi_k - \vartheta_k)), \aleph)\\
&\succcurlyeq \mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}\left(-\frac{|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k|}{\aleph/F}\right).
\end{aligned} \tag{54}$$

Then, there is a unique solution $\mathfrak{f}: \mathfrak{F}_1 \longrightarrow \mathscr{J}$ for (1) such that

$$\begin{aligned}
&\mathfrak{Y}(\mathfrak{h}(\varpi_1 - \vartheta_1) - \mathfrak{f}(\varpi_1 - \vartheta_1), \ldots, \mathfrak{h}(\varpi_k - \vartheta_k) - \mathfrak{f}(\varpi_k - \vartheta_k), \aleph)\\
&\succcurlyeq \mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}\left(-\frac{|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k|}{\aleph(1 - \beth)/E\sum_{\varsigma=0}^{\infty}\left((b-a)^\omega/\omega\right)^{\varsigma\varrho}/\Gamma(1 + \tau\varsigma\varrho)F}\right),
\end{aligned} \tag{55}$$

Proof. We set

$$\mathscr{H} := \{\mathfrak{f}: \mathfrak{F}_1 \longrightarrow \mathscr{J}, \mathfrak{f} \text{ is differentiable}\}, \tag{56}$$

and introduce the $\mathfrak{F}_4$-valued metric on $\mathscr{H}$ as

$$\begin{aligned}
&\inf\{\mathfrak{R} \in \mathfrak{F}_6: \mathscr{N}(\mathfrak{h}(\varpi_1 - \vartheta_1)\\
&\quad - \mathfrak{f}(\varpi_1 - \vartheta_1), \ldots, \mathfrak{h}(\varpi_k - \vartheta_k) - \mathfrak{f}(\varpi_k - \vartheta_k), \aleph)\\
&\succcurlyeq \mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}\left(-\frac{|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k|}{\aleph/\mathfrak{R}}\right),\\
&\forall \mathfrak{f}, \mathfrak{h} \in \mathscr{H}, \varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k \in \mathfrak{F}_1, \aleph \in \mathfrak{F}_2\} = 0.
\end{aligned} \tag{57}$$

By Theorem 4, we have $(\mathscr{H}, d)$ that is a complete $\mathfrak{F}_4$-valued metric space. □

Step 1. We define Π from $\mathscr{H}$ to $\mathscr{H}$ by the following equation:

$$\Pi(\mathfrak{f}(\varpi_i-\vartheta_i))=\mathfrak{f}_a\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_i-\vartheta_i)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}+\frac{1}{\Gamma(\omega)}\int_a^{(\varpi_i-\vartheta_i)}\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_i-\vartheta_i)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\cdot\sum_{\varsigma=0}^{\infty}\frac{((\lambda_i-\lambda_i')-\mathrm{a})^{\omega}/\omega)^{-k\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\left(\frac{(\lambda_i-\lambda_i')}{t}\right)^{s}\left(\log\frac{t}{(\lambda_i-\lambda_i')}\right)^{\omega-1}\frac{K((\lambda_i-\lambda_i'),\mathfrak{f}((\lambda_i-\lambda_i')))}{(\lambda_i-\lambda_i')}d(\lambda_i-\lambda_i'), \tag{58}$$

for $\varpi_i-\vartheta_i\in\mathfrak{F}_1$ $(i=1,2,\ldots,k)$, and we show Π is a strictly contractive mapping.

Let $\mathfrak{f},\mathfrak{h}\in\mathscr{H}$ and consider the coefficient $\mathfrak{R}_{\mathfrak{f}\mathfrak{h}}\in\mathfrak{F}_4$ with $d(\mathfrak{f},\mathfrak{h})\le\mathfrak{R}_{\mathfrak{f}\mathfrak{h}}$; thus, we have

$$\mathscr{N}\left(\mathfrak{f}(\varpi_1-\vartheta_1)-\mathfrak{h}(\varpi_1-\vartheta_1),\ldots,\mathfrak{f}(\varpi_k-\vartheta_k)-\mathfrak{h}(\varpi_k-\vartheta_k),\mathfrak{R}_{\mathfrak{f}\mathfrak{h}}\aleph\right)\succcurlyeq\mathbf{H}_{\tau,\omega}^{\mathbf{m},\mathbf{n}}\left(-\frac{|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|}{\aleph}\right), \tag{59}$$

for all $\mathfrak{f},\mathfrak{h}\in\mathscr{H},\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k\in\mathfrak{F}_1$ and $\aleph\in\mathfrak{F}_2$. Applying (MVFkN2) and (MVFkN3), we imply that

$$\begin{aligned}
&\mathscr{N}\left(\Pi\mathfrak{f}(\varpi_1-\vartheta_1)-\Pi\mathfrak{h}(\varpi_1-\vartheta_1),\ldots,\Pi\mathfrak{f}(\varpi_k-\vartheta_k)-\Pi\mathfrak{h}(\varpi_k-\vartheta_k),\mathfrak{R}_{\mathfrak{f}\mathfrak{h}}\aleph\right)\\
&=\mathscr{N}\Bigg(\mathfrak{f}_a\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_1-\vartheta_1)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}+\frac{1}{\Gamma(\omega)}\int_a^{(\varpi_1-\vartheta_1)}\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_1-\vartheta_1)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\sum_{\varsigma=0}^{\infty}\frac{((\lambda_1-\lambda_i')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\\
&\cdot\left(\frac{(\lambda_1-\lambda_1')}{\mathrm{t}}\right)^{s}\left(\log\frac{\mathrm{t}}{(\lambda_1-\lambda_1')}\right)^{\omega-1}\frac{\mathfrak{k}((\lambda_1-\lambda_1'),\mathfrak{f}((\lambda_1-\lambda_1')))}{(\lambda_1-\lambda_1')}d(\lambda_1-\lambda_1')\\
&-\mathfrak{h}_{\mathrm{a}}\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_1-\vartheta_1)-\mathrm{a})^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}+\frac{1}{\Gamma(\omega)}\int_a^{(\varpi_1-\vartheta_1)}\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_1-\vartheta_1)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\sum_{\varsigma=0}^{\infty}\frac{((\lambda_1-\lambda_1')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\\
&\cdot\left(\frac{(\lambda_1-\lambda_1')}{t}\right)^{s}\left(\log\frac{t}{(\lambda_1-\lambda_1')}\right)^{\omega-1}\frac{K((\lambda_1-\lambda_1'),\mathfrak{h}((\lambda_1-\lambda_1')))}{(\lambda_1-\lambda_1')}d(\lambda_1-\lambda_1'),\\
&\ldots,\mathfrak{f}_{\mathrm{a}}\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}+\frac{1}{\Gamma(\omega)}\int_a^{(\varpi_k-\vartheta_k)}\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\sum_{\varsigma=0}^{\infty}\frac{((\lambda_k-\lambda_k')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\\
&\cdot\left(\frac{(\lambda_k-\lambda_k')}{\mathrm{t}}\right)^{s}\left(\log\frac{t}{(\lambda_k-\lambda_k')}\right)^{\omega-1}\frac{K((\lambda_k-\lambda_k'),\mathfrak{f}((\lambda_k-\lambda_k')))}{(\lambda_k-\lambda_k')}d(\lambda_k-\lambda_k')\\
&-\mathfrak{h}_a\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}+\frac{1}{\Gamma(\omega)}\int_a^{(\varpi_k-\vartheta_k)}\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\sum_{\varsigma=0}^{\infty}\frac{((\lambda_k-\lambda_k')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\\
&\cdot\left(\frac{(\lambda_k-\lambda_k')}{\mathrm{t}}\right)^{s}\left(\log\frac{t}{(\lambda_k-\lambda_k')}\right)^{\omega-1}\frac{K((\lambda_k-\lambda_k'),\mathfrak{h}((\lambda_k-\lambda_k')))}{(\lambda_k-\lambda_k')}d(\lambda_k-\lambda_k'),\mathfrak{R}_{\mathfrak{f}\mathfrak{h}}\aleph\Bigg)\\
&\succcurlyeq\mathscr{N}\Bigg(\frac{1}{\Gamma(\omega)}\int_a^{(\varpi_1-\vartheta_1)}\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_1-\vartheta_1)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\sum_{\varsigma=0}^{\infty}\frac{((\lambda_1-\lambda_1')-\mathfrak{a})^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}
\end{aligned}$$

$$\left(\frac{(\lambda_1-\lambda_1')}{t}\right)^s\left(\log\frac{t}{(\lambda_1-\lambda_1')}\right)^{\omega-1}\frac{K((\lambda_1-\lambda_1'),\mathfrak{f}((\lambda_1-\lambda_1')))-K((\lambda_1-\lambda_1'),\mathfrak{h}((\lambda_1-\lambda_1')))}{(\lambda_1-\lambda_1')}d(\lambda_1-\lambda_1'),\ldots,$$

$$\frac{1}{\Gamma(\omega)}\int_a^{(\varpi_k-\vartheta_k)}\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\sum_{\varsigma=0}^{\infty}\frac{((\lambda_k-\lambda_k')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\left(\frac{(\lambda_k-\lambda_k')}{t}\right)^s\left(\log\frac{\mathrm{t}}{(\lambda_k-\lambda_k')}\right)^{\omega-1}$$

$$\frac{K((\lambda_k-\lambda_k'),\mathfrak{f}((\lambda_k-\lambda_k')))-K((\lambda_k-\lambda_k'),\mathfrak{h}((\lambda_k-\lambda_k')))}{(\lambda_k-\lambda_k')}d(\lambda_k-\lambda_k'),\mathfrak{R}_{\mathfrak{f}\mathfrak{h}}\aleph\Bigg)$$

$$\succcurlyeq\mathcal{N}\left(\frac{1}{\Gamma(\omega)}\int_a^{(\varpi_1-\vartheta_1)}\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_1-\vartheta_1)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\sum_{\varsigma=0}^{\infty}\frac{((\lambda_1-\lambda_1')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\right.$$

$$\left(\frac{(\lambda_1-\lambda_1')}{t}\right)^s\left(\log\frac{t}{(\lambda_1-\lambda_1')}\right)^{\omega-1}\frac{\mathfrak{f}(\lambda_1-\lambda_1')-\mathfrak{h}(\lambda_1-\lambda_1')}{(\lambda_1-\lambda_1')}d(\lambda_1-\lambda_1')$$

$$\ldots,\frac{1}{\Gamma(\zeta)}\int_a^{(\varpi_k-\vartheta_k)}\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\sum_{\varsigma=0}^{\infty}\frac{((\lambda_k-\lambda_k')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\left(\frac{(\lambda_k-\lambda_k')}{t}\right)^s\left(\log\frac{t}{(\lambda_k-\lambda_k')}\right)^{\omega-1}$$

$$\left.\frac{\mathfrak{f}(\lambda_k-\lambda_k')-\mathfrak{h}(\lambda_k-\lambda_k')}{(\lambda_k-\lambda_k')}d(\lambda_k-\lambda_k'),\frac{\mathfrak{R}_{\mathfrak{f}\mathfrak{h}}\mu}{\wp}\right)$$

$$\succcurlyeq\mathbf{H}_{\tau,\omega}^{\mathbf{m},\mathbf{n}}\left(|\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_1-\vartheta_1)-a)^{\omega}/\omega)^{\varsigma\lambda}}{\Gamma(1+\alpha\varsigma\varrho)}\frac{1}{\Gamma(\omega)}\int_a^{(\varpi_1-\vartheta_1)}\sum_{\varsigma=0}^{\infty}\frac{((\lambda_1-\lambda_1')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\right.$$

$$\left(\frac{(\lambda_1-\lambda_1')}{t}\right)^s\left(\log\frac{t}{(\lambda_1-\lambda_1')}\right)^{\omega-1}\frac{d(\lambda_1-\lambda_1')}{\lambda_1-\lambda_1'},$$

$$\ldots,\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\frac{1}{\Gamma(\omega)}\int_{\mathrm{a}}^{(\varpi_k-\vartheta_k)}\sum_{\varsigma=0}^{\infty}\frac{((\lambda_k-\lambda_k')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\left(\frac{(\lambda_k-\lambda_k')}{t}\right)^s\left(\log\frac{t}{(\lambda_k-\lambda_k')}\right)^{\omega-1}$$

$$\left.\frac{d(\lambda_k-\lambda_k')}{(\lambda_k-\lambda_k')}|\frac{1}{\aleph/\wp}\right)$$

$$\succcurlyeq\mathbf{H}_{\tau,\omega}^{\mathbf{m},\mathbf{n}}\left(|\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_1-\vartheta_1)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)},\ldots,\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\frac{1}{\Gamma(\omega)}\left[\int_a^{(\varpi_1-\vartheta_1)}\sum_{\varsigma=0}^{\infty}\frac{((\lambda_1-\lambda_1')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\right.\right.$$

$$,\ldots,\int_a^{(\varpi_k-\vartheta_k)}\sum_{\varsigma=0}^{\infty}\frac{((\lambda_k-\lambda_k')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\left(\frac{(\lambda_k-\lambda_k')}{t}\right)^s\left(\log\frac{t}{(\lambda_k-\lambda_k')}\right)^{\omega-1}$$

$$\left.\left.\frac{d(\lambda_k-\lambda_k')}{(\lambda_k-\lambda_k')}\right]|\frac{1}{\aleph/\wp}\right)$$

$$\succcurlyeq\mathbf{H}_{\tau,\omega}^{\mathbf{m},\mathbf{n}}\left(-\frac{|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|}{\aleph/\wp\left(\sum_{\varsigma=0}^{\infty}\sum_{\varsigma_1\cdots\varsigma_k=\varsigma}^{\infty}\begin{pmatrix}\varsigma\\ \varsigma_1,\ldots,\varsigma_k\end{pmatrix}(((\varpi_1-\vartheta_1)-a)^{\omega}/\omega)^{\varsigma_1\varrho}\cdots(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma_k\varrho}/\Gamma(\varsigma_1\cdots\varsigma_k+\tau\varsigma\varrho)\right)E}\right),\qquad(60)$$

which implies that

$$d(\Pi(f),\Pi(\mathfrak{h}))\leq \wp\left(\sum_{\varsigma=0}^{\infty}\sum_{\varsigma_1\cdots\varsigma_k=\varsigma}^{\infty}\binom{\varsigma}{\varsigma_1,\ldots,\varsigma_k}\frac{(((\varpi_1-\vartheta_1)-a)^{\omega}/\omega)^{\varsigma_1\varrho}\cdots(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma_k\varrho}}{\Gamma(\varsigma_1\cdots\varsigma_k+\tau\varsigma\varrho)}\right)E\mathfrak{R}_{\mathfrak{f}\mathfrak{h}}, \tag{61}$$

so

$$d(\Pi(f),\Pi(\mathfrak{h}))\leq \wp\left(\sum_{\varsigma=0}^{\infty}\sum_{\varsigma_1\cdots\varsigma_k=\varsigma}^{\infty}\binom{\varsigma}{\varsigma_1,\ldots,\varsigma_k}\frac{(((\varpi_1-\vartheta_1)-a)^{\omega}/\omega)^{\varsigma_1\varrho}\cdots(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma_k\varrho}}{\Gamma(\varsigma_1\cdots\varsigma_k+\tau\varsigma\varrho)}\right)\mathrm{E}d(f,\mathfrak{h}), \tag{62}$$

where $0<\wp\left(\sum_{\varsigma=0}^{\infty}\sum_{\varsigma_1\cdots\varsigma_k=\varsigma}^{\infty}\binom{\varsigma}{\varsigma_1,\ldots,\varsigma_k}(((\varpi_1-\vartheta_1)-\mathfrak{c}c)^{\omega}/\omega)^{\varsigma_1\varrho}\cdots(((\varpi_k-\vartheta_k)-\mathfrak{c}c)^{\omega}/\omega)^{\varsigma_k\varrho}/\Gamma(\varsigma_1\cdots\varsigma_k+\tau\varsigma\varrho)\right)E<1$; therefore, Π is a contraction mapping.

Step 2. We will show that $d(\Pi(\mathfrak{h}),\mathfrak{h})<\infty$.

Let $\mathfrak{h}\in\mathscr{H}$, we have

$$\begin{aligned}
&\mathcal{N}(\Pi(\mathfrak{h}(\varpi_1-\vartheta_1))-\mathfrak{h}(\varpi_1-\vartheta_1),\ldots,\Pi(\varpi_k-\vartheta_k))-\mathfrak{h}(\varpi_k-\vartheta_k),\aleph)\\
&=\mathcal{N}\Bigg(\mathfrak{h}_a\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_1-\vartheta_1)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}+\frac{1}{\Gamma(\omega)}\int_a^{(\varpi_1-\vartheta_1)}\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_1-\vartheta_1)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\sum_{\varsigma=0}^{\infty}\frac{((\lambda_1-\lambda_1')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\\
&\left(\frac{(\lambda_1-\lambda_1')}{t}\right)^{s}\left(\log\frac{t}{(\lambda_1-\lambda_1')}\right)^{\omega-1}\frac{K((\lambda_1-\lambda_1'),\mathfrak{h}((\lambda_1-\lambda_1')))}{(\lambda_1-\lambda_1')}d(\lambda_1-\lambda_1')-\mathfrak{h}(\varpi_1-\vartheta_1),\ldots,\\
&\mathfrak{h}_a\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}+\frac{1}{\Gamma(\omega)}\int_a^{(\varpi_k-\vartheta_k)}\sum_{\varsigma=0}^{\infty}\frac{(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\sum_{\varsigma=0}^{\infty}\frac{((\lambda_k-\lambda_k')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\\
&\left(\frac{(\lambda_k-\lambda_k')}{t}\right)^{s}\left(\log\frac{t}{(\lambda_k-\lambda_k')}\right)^{\omega-1}\frac{K((\lambda_k-\lambda_k'),\mathfrak{h}((\lambda_k-\lambda_k')))}{(\lambda_k-\lambda_k')}d(\lambda_k-\lambda_k')-\mathfrak{h}(\varpi_k-\vartheta_k),\aleph\Bigg)\\
&\succcurlyeq\mathbf{H}^{\mathbf{m},\mathbf{n}}_{\tau,\omega}\Bigg(|\sum_{\varsigma=0}^{\infty}\frac{((b-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\Bigg[\frac{1}{\Gamma(\beta)}\int_a^{(\varpi_1-\vartheta_1)}\sum_{\varsigma=0}^{\infty}\frac{(((\lambda_1-\lambda_1')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\left(\frac{(\lambda_1-\lambda_1')}{t}\right)^{s}\left(\log\frac{t}{(\lambda_1-\lambda_1')}\right)^{\omega-1}\\
&\frac{d(\lambda_1-\lambda_1')}{(\lambda_1-\lambda_1')},\ldots,\frac{1}{\Gamma(\omega)}\int_a^{(\varpi_k-\vartheta_k)}\sum_{\varsigma=0}^{\infty}\frac{(((\lambda_k-\lambda_k')-a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\left(\frac{(\lambda_k-\lambda_k')}{t}\right)^{s}\left(\log\frac{t}{(\lambda_k-\lambda_k')}\right)^{\omega-1}\\
&\frac{d(\lambda_k-\lambda_k')}{(\lambda_k-\lambda_k')}\Bigg]|\frac{1}{\aleph}\Bigg)\\
&\succcurlyeq\mathbf{H}^{\mathbf{m},\mathbf{n}}_{\tau,\omega}\left(-\frac{|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|}{\aleph/\mathsf{F}\sum_{\varsigma=0}^{\infty}((b-a)^{\omega}/\omega)^{\varsigma\varrho}/\Gamma(1+\tau\varsigma\varrho)E}\right).
\end{aligned}\tag{63}$$

Consequently, we obtain the following equation:

$$d(\Pi\mathfrak{h},\mathfrak{h})\leq\mathsf{F}\sum_{\varsigma=0}^{\infty}\frac{((b-a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}E<\infty, \tag{64}$$

for every $\aleph\in\mathfrak{F}_2$ and $\mathrm{E}<1$. Then, we have $d(\Pi(\mathfrak{h}),\mathfrak{h})<\infty$.

Therefore, all the conditions of Theorem 1 hold. Then, we have

(1) The sequence $\{\Pi^{e}\mathfrak{f}\}$ converges to a fixed point such as $\mathfrak{f}$.

(2) The unique element $\mathfrak{f}$ is in the set $\mathscr{H}^{*} = \{\mathfrak{h} \in \mathscr{H}\colon d(\Pi\mathfrak{h}, \mathfrak{f}) < \infty\}$ and is the unique fixed point of Π, which means $\Pi\mathfrak{f} = \mathfrak{f}$ or equivalently as shown in the following equation:

$$\begin{aligned}\mathfrak{f}(\varpi_i - \vartheta_i) = \mathfrak{f}_a \sum_{\varsigma=0}^{\infty} \frac{(((\varpi_i - \vartheta_i) - a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)} + \frac{1}{\Gamma(\omega)} \int_a^{(\varpi_i - \vartheta_i)} \sum_{\varsigma=0}^{\infty} \frac{(((\varpi_i - \vartheta_i) - a)^{\omega}/\omega)^{\varsigma\varrho}}{\Gamma(1+\tau\varsigma\lambda)} \sum_{\varsigma=0}^{\infty} \frac{((\varpi_i - \vartheta_i) - a)^{\omega}/\omega)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)} \\ \cdot \left(\frac{(\lambda_i - \lambda_i')}{t}\right)^{s} \left(\log \frac{t}{(\lambda_i - \lambda_i')}\right)^{\omega-1} \frac{K((\lambda_i - \lambda_i'), \mathfrak{f}((\lambda_i - \lambda_i')))}{(\lambda_i - \lambda_i')} d(\lambda_i - \lambda_i'),\end{aligned} \quad (65)$$

where $i = 1, \ldots, k$.

Since $\mathfrak{f}$ is a differentiable function, by the CFD and according to (65) and Lemma 1, we have

$$\mathfrak{D}_{\omega}^{a}\mathfrak{f}(\varpi - \vartheta) = \varrho\mathfrak{f}(\varpi - \vartheta) + K(\varpi - \vartheta, \mathfrak{f}(\varpi - \vartheta)). \quad (66)$$

(3) Using inequality (64), we get

$$\begin{aligned} d(\mathfrak{w}, \mathfrak{f}) &\le \frac{1}{1-\beth} d(\Pi\mathfrak{h}, \mathfrak{h}) \\ &\le \frac{\digamma \sum_{\varsigma=0}^{\infty} ((\mathfrak{d} - \mathrm{a})^{\omega}/\omega)^{\varsigma\varrho}/\Gamma(1+\tau\varsigma\varrho)E}{1-\beth}. \end{aligned} \quad (67)$$

Thus, equation (1) has the Hyers–Ulam-H-Fox stability property.

Now, we show the uniqueness of the obtained point. For convenience, we consider the following equation:

$$\epsilon = \frac{\digamma \sum_{\varsigma=0}^{\infty} (b-a)^{\omega}/\omega^{\varsigma\varrho}/\Gamma(1+\tau\varsigma\varrho)E}{1-\beth}, \quad (68)$$

and let $\mathfrak{g}$ be another differentiable function satisfying equation (66), and this means that the following equation holds:

$$\mathfrak{D}_{\omega}^{a}\mathfrak{g}(\varpi - \vartheta) = \varrho\mathfrak{f}(\varpi - \vartheta) + K(\varpi - \vartheta, \mathfrak{g}(\varpi - \vartheta)). \quad (69)$$

We are ready to prove that $\mathfrak{g}$ is a fixed point of Π and $\mathfrak{g} \in \mathscr{H}^{*}$. Using equation (69), we get $\Pi\mathfrak{g} = \mathfrak{g}$. Now, we show that $d(\Pi\mathfrak{h}, \mathfrak{g}) < \infty$. Let $\mathfrak{h} \in \mathscr{H}$, $d(\mathfrak{h}, \mathfrak{g}) < \epsilon$, and from equation (69), we get

$$\begin{aligned}&\frac{1}{\aleph/\wp\left(\sum_{\varsigma=0}^{\infty}\sum_{\varsigma_1\cdots\varsigma_k=\varsigma}\binom{\varsigma}{\varsigma_1,\ldots,\varsigma_k}(((\varpi_1-\vartheta_1)-\mathrm{a})^{\omega}/\omega)^{\varsigma_1\tilde{n}}\cdots(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma_k\tilde{n}}/\Gamma(\varsigma_1\cdots\varsigma_k+\tau\varsigma\tilde{n})\right)\grave{o}} \\ &\succcurlyeq \mathbf{H}_{\tau,\omega}^{\mathbf{m,n}}\left(-\frac{|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|}{\aleph/\wp\left(\sum_{\varsigma=0}^{\infty}\sum_{\varsigma_1\cdots\varsigma_k=\varsigma}\binom{\varsigma}{\varsigma_1,\ldots,\varsigma_k}(((\varpi_1-\vartheta_1)-a)^{\omega}/\omega)^{\varsigma_1\tilde{n}}\cdots(((\varpi_k-\vartheta_k)-a)^{\omega}/\omega)^{\varsigma_k\tilde{n}}/\Gamma(\varsigma_1\cdots\varsigma_k+\tau\varsigma\tilde{n})\right)\grave{o}\mathrm{E}}\right),\end{aligned} \quad (70)$$

Then, we have

$$d(\Pi\mathrm{h}, \mathrm{g}) \le \left[\wp\left(\sum_{\varsigma=0}^{\infty}\sum_{\varsigma_1\cdots\varsigma_k=\varsigma}\binom{\varsigma}{\varsigma_1,\ldots,\varsigma_k}(((\varpi_1-\vartheta_1)-a)^{\omega}/\omega)^{\varsigma_1\tilde{n}}\cdots\left(\frac{((\varpi_k-\vartheta_k)-a)^{\omega}}{\omega}\right)^{\varsigma_k\tilde{n}}/\Gamma(\varsigma_1\cdots\varsigma_k+\tau\varsigma\tilde{n})\right)E\right]\epsilon < \infty. \quad (71)$$

4. Example

Now, we provide numerical examples according to the results obtained.

Example 1. We consider the following conformable FDE:

$$\begin{cases} \mathfrak{D}^1_{1/2}\mathfrak{f}(\varpi-\vartheta) = \mathfrak{f}(\varpi-\vartheta) + \dfrac{1}{5+(\varpi-\vartheta)^2}\left(\dfrac{\mathfrak{f}^2(\varpi-\vartheta)}{\mathfrak{f}(\varpi-\vartheta)+1}\right)\cos(\mathfrak{f}(\varpi-\vartheta)),\ (\varpi-\vartheta)\in(1,2] \\ \mathfrak{f}(1)=1 \end{cases}, \quad (72)$$

where $\omega = 1/2, \varrho = 1, \wp = 1/8$, for $\varsigma = 1, (\varpi_1 - \vartheta_1) = 2$, $\tau = 1$ we have $\beth = 0/4618 < 1$. Also, in this equation, $K(\varpi - \vartheta, \mathfrak{f}(\varpi-\vartheta)) = 1/5 + (\varpi-\vartheta)^2 (\mathfrak{f}^2(\varpi-\vartheta)/\ \mathfrak{f}(\varpi-\vartheta)+1)\ \cos(\mathfrak{f}(\varpi-\vartheta))$. Let for mapping $\mathfrak{k}$ and the MVF control function $\mathbf{H}^{\mathbf{m},\mathbf{n}}_{\tau,\omega}$, we have

(1)

$$\begin{aligned} &\mathcal{N}(K((\varpi_1-\vartheta_1),\mathfrak{f}((\varpi_1-\vartheta_1))) - K((\varpi_1-\vartheta_1),\mathfrak{h}((\varpi_1-\vartheta_1))),\ldots, \\ &K((\varpi_k-\vartheta_k),\mathfrak{f}((\varpi_k-\vartheta_k))) - K((\varpi_k-\vartheta_k),\mathfrak{h}((\varpi_k-\vartheta_k))),\aleph) \\ &\succcurlyeq \mathcal{N}\left(\mathfrak{f}((\varpi_1-\vartheta_1)) - \mathfrak{h}((\varpi_1-\vartheta_1)),\ldots,\mathfrak{f}((\varpi_k-\vartheta_k)) - \mathfrak{h}((\varpi_k-\vartheta_k)),\frac{\aleph}{1/8}\right). \end{aligned} \quad (73)$$

(2)

$$\begin{aligned} &\mathbf{H}^{\mathbf{m},\mathbf{n}}_{\tau,\omega}|\left(\frac{1}{\Gamma(1/2)}\left[\int_a^{(\varpi_1-\vartheta_1)}\sum_{\varsigma=0}^{\infty}\frac{((2(\lambdabar_1-\lambdabar_1')-a)^{1/2})^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\left(\frac{(\lambdabar_1-\lambdabar_1')}{t}\right)^s\left(\log\frac{t}{(\lambdabar_1-\lambdabar_1')}\right)^{-1/2}\frac{d(\lambdabar_1-\lambdabar_1')}{(\lambdabar_1-\lambdabar_1')},\right.\right. \\ &\left.\left.\ldots,\int_a^{(\varpi_k-\vartheta_k)}\sum_{\varsigma=0}^{\infty}\frac{(2(\lambdabar_k-\lambdabar_k')-a)^{1/2})^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\left(\frac{(\lambdabar_k-\lambdabar_k')}{t}\right)^s\left(\log\frac{t}{(\lambdabar_k-\lambdabar_k')}\right)^{\omega-1}\frac{d(\lambdabar_k-\lambdabar_k')}{(\lambdabar_k-\lambdabar_k')}\right]\frac{1}{\aleph}|\right) \\ &\succcurlyeq \mathbf{H}^{\mathbf{m},\mathbf{n}}_{\tau,\omega}\left(-\frac{|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|}{\aleph/1/2}\right). \end{aligned} \quad (74)$$

If $\mathfrak{h}\in C([1,2],\mathbb{R})$ be a differentiable function such that

$$\begin{aligned} &\mathcal{N}\mathfrak{D}^1_{1/2}\mathfrak{h}(\varpi_1-\vartheta_1) - \mathfrak{h}(\varpi_1-\vartheta_1) - \frac{1}{5+(\varpi_1-\vartheta_1)^2}\left(\frac{\mathfrak{h}^2(\varpi_1-\vartheta_1)}{\mathfrak{h}(\varpi_1-\vartheta_1)+1}\right)\cos(\mathfrak{h}(\varpi_1-\vartheta_1)),\ldots, \\ &\left.\mathfrak{D}^1_{1/2}\mathfrak{h}(\varpi_k-\vartheta_k) - \mathfrak{h}(\varpi_k-\vartheta_k) - \frac{1}{5+(\varpi_k-\vartheta_k)^2}\left(\frac{\mathfrak{h}^2(\varpi_k-\vartheta_k)}{\mathfrak{h}(\varpi_k-\vartheta_k)+1}\right)\cos(\mathfrak{h}(\varpi_k-\vartheta_k)),\aleph\right) \\ &\succcurlyeq \mathbf{H}^{\mathbf{m},\mathbf{n}}_{\tau,\omega}\left(-\frac{|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|}{\aleph/\digamma}\right), \end{aligned} \quad (75)$$

then $\mathfrak{h}$ is a solution of the inequality

$$\mathcal{N}\Bigg(h(\varpi_1-\vartheta_1)-h_a\sum_{\varsigma=0}^{\infty}\frac{\left(4(\varpi_1-\vartheta_1)-a)^{1/4}\right)^{\varsigma}}{\Gamma(1+\tau\varsigma)} -\frac{1}{\Gamma(1/4)}\int_a^{(\varpi_1-\vartheta_1)}\sum_{\varsigma=0}^{\infty}\frac{\left(4(\varpi_1-\vartheta_1)-a)^{1/4}\right)^{\varsigma}}{\Gamma(1+\tau\varsigma)}\sum_{\varsigma=0}^{\infty}\frac{\left((4(\lambda_1-\lambda_1')-a)^{1/4}\right)^{-\varsigma}}{\Gamma(1+\tau\varsigma)} \cdot\frac{\sqrt{(\lambda_1-\lambda_1')+1}}{6(\lambda_1-\lambda_1')^2+6}\sin(h(\lambda_1-\lambda_1'))\left(\frac{(\lambda_1-\lambda_1')}{t}\right)^s\left(\log\frac{t}{(\lambda_1-\lambda_1')}\right)^{-3/4}\frac{d(\lambda_1-\lambda_1')}{(\lambda_1-\lambda_1')},\ldots, \cdot h(\varpi_k-\vartheta_k)-\mathfrak{f}_a\sum_{\varsigma=0}^{\infty}\frac{\left((4(\varpi_k-\vartheta_k)-a)^{1/4}\right)^{\varsigma}}{\Gamma(1+\tau\varsigma)}-\frac{1}{\Gamma(1/4)}\int_a^{(\varpi_k-\vartheta_k)}\sum_{\varsigma=0}^{\infty}\frac{\left((4(\varpi_k-\vartheta_k)-a)^{1/4}\right)^{\varsigma}}{\Gamma(1+\tau\varsigma)} \cdot\sum_{\varsigma=0}^{\infty}\frac{\left((4(\lambda_k-\lambda_k')-\text{textgothc})^{1/4}\right)^{-\varsigma}}{\Gamma(1+\tau\varsigma)}\frac{\sqrt{(\lambda_k-\lambda_k')+1}}{6(\lambda_k-\lambda_k')^2+6}\sin(\mathfrak{h}(\lambda_k-\lambda_k')) \cdot\left(\frac{(\lambda_k-\lambda_k')}{t}\right)^s\left(\log\frac{t}{(\lambda_k-\lambda_k')}\right)^{-3/4}\frac{d(\lambda_k-\lambda_k')}{(\lambda_k-\lambda_k')},\aleph\Bigg) \succcurlyeq \mathbf{H}_{\tau,\omega}^{\mathbf{m},\mathbf{n}}\Bigg|\left(\frac{1}{\Gamma(1/4)}\int_a^{(\varpi_1-\vartheta_1)}\sum_{\varsigma=0}^{\infty}\frac{\left((4(\lambda_1-\lambda_1')-a)^{1/4}\right)^{-\varsigma}}{\Gamma(1+\tau\varsigma)}\right)\left(\frac{(\lambda_1-\lambda_1')}{t}\right)^s\left(\log\frac{t}{(\lambda_1-\lambda_1')}\right)^{-3/4}\frac{d(\lambda_1-\lambda_1')}{(\lambda_1-\lambda_1')},\ldots, \cdot\frac{1}{\Gamma(1/4)}\int_a^{(\varpi_k-\vartheta_k)}\sum_{\varsigma=0}^{\infty}\frac{\left((4(\lambda_k-\lambda_k')-a)^{1/4}\right)^{-\varsigma}}{\Gamma(1+\tau\varsigma)}\left(\frac{(\lambda_k-\lambda_k')}{t}\right)^s\left(\log\frac{t}{(\lambda_k-\lambda_k')}\right)^{-3/4}\frac{d(\lambda_k-\lambda_k')}{(\lambda_k-\lambda_k')}\Big|\frac{1}{\aleph/1/3\sum_{\varsigma=0}^{\infty}(4)^{\varsigma}/\Gamma(1+\tau\varsigma)}\Bigg), \tag{76}$$

and thus, we can find a unique differentiable function $\mathfrak{f}\in C([1,2],\mathbb{R})$ from (72) such that for each $(\varpi-\vartheta)\in[1,2]$, we have

$$\mathfrak{f}(\varpi_i-\vartheta_i)=\sum_{\varsigma=0}^{\infty}\frac{\left(2((\varpi_i-\vartheta_i)-1)^{1/2}\right)^{\varsigma}}{\Gamma(1+\varsigma\tau)} +\frac{1}{\Gamma(1/2)}\int_1^{(\varpi_i-\vartheta_i)}\sum_{\varsigma=0}^{\infty}\frac{\left(2((\varpi_i-\vartheta_i)-1)^{1/2}\right)^{\varsigma}}{\Gamma(1+\varsigma\tau)}\sum_{\varsigma=0}^{\infty}\frac{\left(2(\lambda\!\!\!^{-}_i-\lambda\!\!\!^{-}_i{}'-1)^{1/2}\right)^{-\varsigma}}{\Gamma(1+\varsigma\tau)}\left(\frac{\lambda\!\!\!^{-}_i-\lambda\!\!\!^{-}_i{}'}{\mathrm{t}}\right)^s\left(\log\frac{t}{\lambda\!\!\!^{-}_i-\lambda\!\!\!^{-}_i{}'}\right)^{-1/2} \frac{1/5+(\varpi_i-\vartheta_i)^2\left(\mathfrak{f}^2(\varpi_i-\vartheta_i)/\mathfrak{f}((\varpi_i-\vartheta_i)+1\right)\cos(\mathfrak{f}(\varpi_i-\vartheta_i))}{(\lambda\!\!\!^{-}_i-\lambda\!\!\!^{-}_i{}')}d(\lambda\!\!\!^{-}_i-\lambda\!\!\!^{-}_i{}'), \tag{77}$$

for any $i=1,\ldots,k$. Therefore

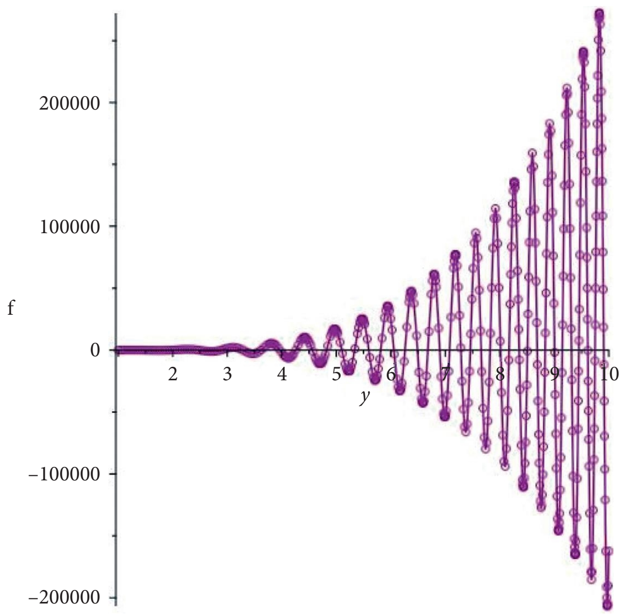

Figure 1: Exact solution of Example 1 for $(\varpi - \vartheta) \in [1, 10]$.

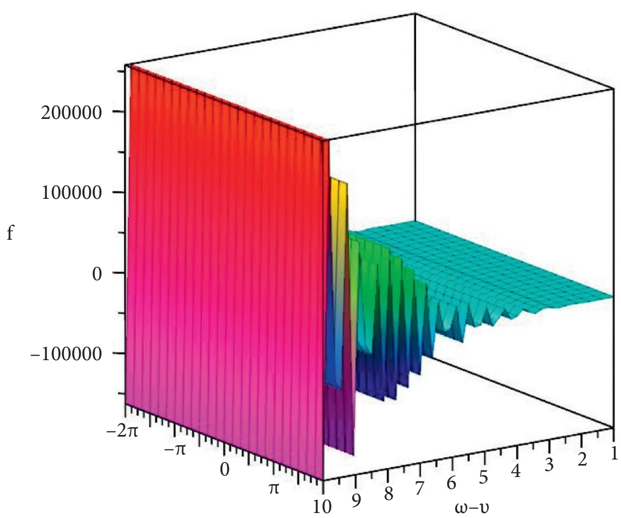

Figure 2: Exact solution of Example 1 for $(\varpi - \vartheta) \in [1, 10]$.

$$d(\mathfrak{h}, \mathfrak{f}) \leq \frac{2}{1 - \beth} \sum_{\varsigma=0}^{\infty} \frac{2^{\varsigma}}{\Gamma(1 + \tau\varsigma)} \mathrm{F},$$

$$\mathcal{N}\left(\mathfrak{h}(\varpi_1 - \vartheta_1) - \mathfrak{f}(\varpi_1 - \vartheta_1), \ldots, \mathfrak{h}(\varpi_k - \vartheta_k) - \mathfrak{f}(\varpi_k - \vartheta_k), \aleph\right)$$

$$\succcurlyeq \mathbf{H}_{\tau,\omega}^{\mathbf{m},\mathbf{n}}\left(-\frac{|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k|}{\aleph(1 - \beth)/2 \sum_{\varsigma=0}^{\infty} 2^{\varsigma}/\Gamma(1 + \tau\varsigma)\mathrm{F}}\right). \tag{78}$$

In Figures 1–8, the exact solution of conformable FDE (72) for $\tau = 1/2, s = 1, t = 1/4$ is demonstrated.

Example 2. We consider the following conformable FDE:

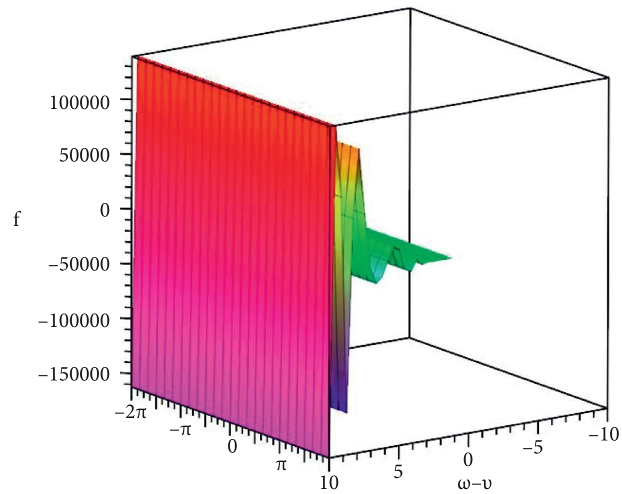

Figure 3: Exact solution of Example 1 for $(\varpi - \vartheta) \in [-10, 10]$.

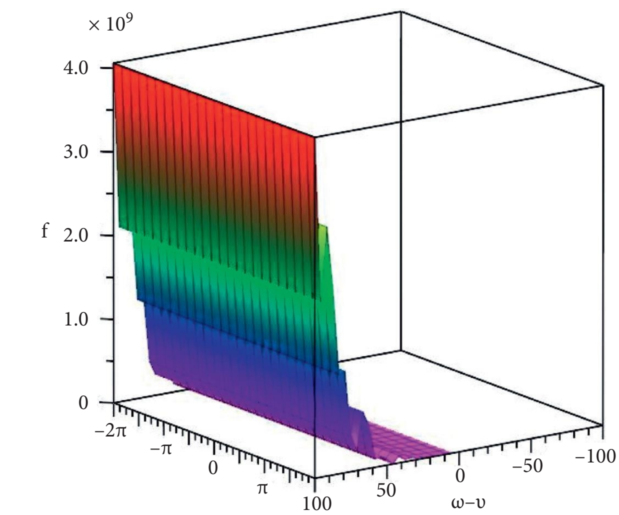

Figure 4: Exact solution of Example 1 for $(\varpi - \vartheta) \in [-100, 100]$.

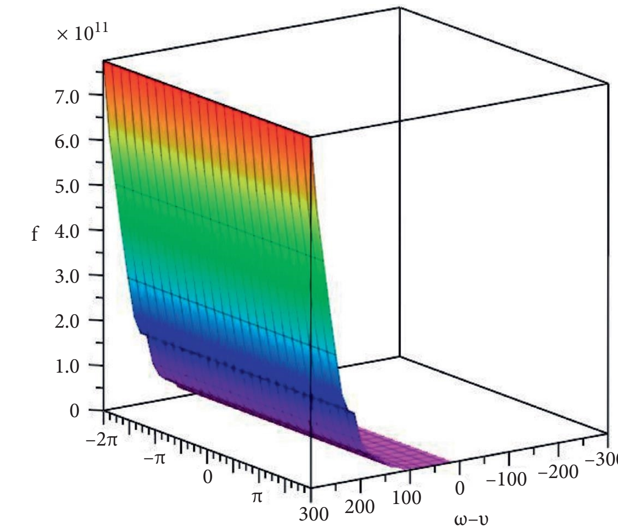

Figure 5: Exact solution of Example 1 for $(\varpi - \vartheta) \in [-300, 300]$.

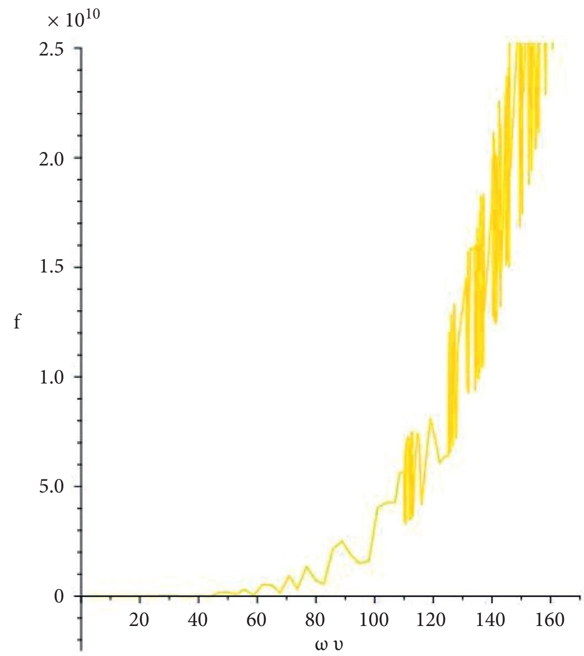

Figure 6: Exact solution of Example 1 for $(\varpi-\vartheta)\in[-300,300]$.

$$\begin{cases} \mathfrak{D}^{1}_{1/4}\mathfrak{f}(\varpi-\vartheta) = \mathfrak{f}(\varpi-\vartheta) + \dfrac{\sqrt{(\varpi-\vartheta)+1}}{6(\varpi-\vartheta)^2+6}\sin(\mathfrak{f}(\varpi-\vartheta)),\ (\varpi-\vartheta)\in(1,2] \\ \mathfrak{f}(1)=1 \end{cases}, \tag{79}$$

where $\omega == 1/4, \varrho = 1, \wp = 1/100$, for $\varsigma = 2$, $(\varpi_1-\vartheta_1) = 4$, $\tau = 2$ we have $\sqsupset = 0.004274 < 1$. Also, in this equation, $\mathfrak{k}(\varpi-\vartheta, \mathfrak{f}(\varpi-\vartheta)) = \sqrt{(\varpi-\vartheta)+1}/6(\varpi-\vartheta)^2+6\sin(\mathfrak{f}(\varpi-\vartheta))$. Let for mapping $\mathfrak{k}$ and the MVF control function $\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}$, we have

$$\begin{aligned} &\mathcal{N}(K((\varpi_1-\vartheta_1),\mathfrak{f}((\varpi_1-\vartheta_1)))-K((\varpi_1-\vartheta_1),\mathfrak{h}((\varpi_1-\vartheta_1))),\ldots,\\ &K((\varpi_k-\vartheta_k),\mathfrak{f}((\varpi_k-\vartheta_k)))-K((\varpi_k-\vartheta_k),\mathfrak{h}((\varpi_k-\vartheta_k))),\aleph)\\ &\succcurlyeq\mathcal{N}(\mathfrak{f}((\varpi_1-\vartheta_1))-\mathfrak{h}((\varpi_1-\vartheta_1)),\ldots,\mathfrak{f}((\varpi_k-\vartheta_k))-\mathfrak{h}((\varpi_k-\vartheta_k)),100\aleph). \end{aligned} \tag{80}$$

$$\begin{aligned} &\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}\Bigg(\Big|\frac{1}{\Gamma(1/4)}\Bigg[\int_{a}^{(\varpi_1-\vartheta_1)}\sum_{\varsigma=0}^{\infty}\frac{\left((4(\lambda_1-\lambda_1')-a)^{1/4}\right)^{-\varsigma\varrho}}{\Gamma(1+\tau\varsigma\varrho)}\left(\frac{(\lambda_1-\lambda_1')}{t}\right)^{s}\left(\log\frac{t}{(\xi_1-\xi_1')}\right)^{-3/4}\frac{d(\lambda_1-\lambda_1')}{(\lambda_1-\lambda_1')},\\ &\ldots,\int_{a}^{(\varpi_k-\vartheta_k)}\sum_{\varsigma=0}^{\infty}\frac{\left(4(\lambda_k-\lambda_k')-a\right)^{1/4^{-\varsigma\varrho}}}{\Gamma(1+\tau\varsigma\varrho)}\left(\frac{(\lambda_k-\lambda_k')}{t}\right)^{s}\left(\log\frac{t}{(\lambda_k-\lambda_k')}\right)^{-3/4}\frac{d(\lambda_k-\lambda_k')}{(\lambda_k-\lambda_k')}\Bigg]\frac{1}{\aleph}\Big|\Bigg)\\ &\succcurlyeq\mathbf{H}^{\mathbf{m,n}}_{\tau,\omega}\left(-\frac{|\varpi_1-\vartheta_1,\ldots,\varpi_k-\vartheta_k|}{4\aleph}\right). \end{aligned} \tag{81}$$

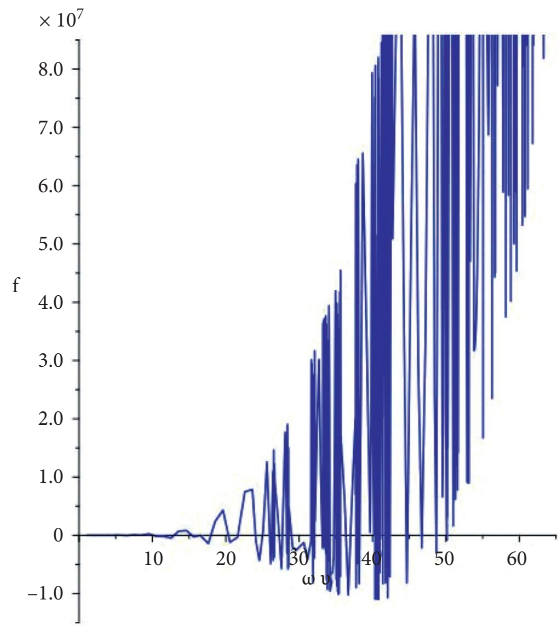

Figure 7: Exact solution of Example 1 for $(\varpi - \vartheta) \in [-100, 100]$.

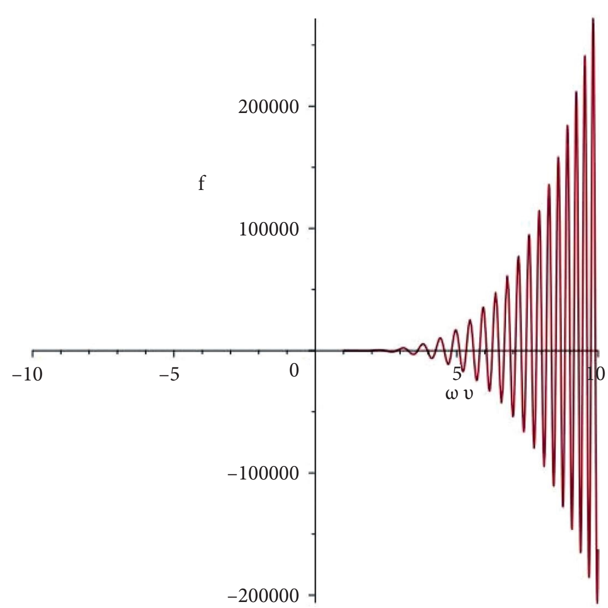

Figure 8: Exact solution of Example 1 for $(\varpi - \vartheta) \in [-10, 10]$.

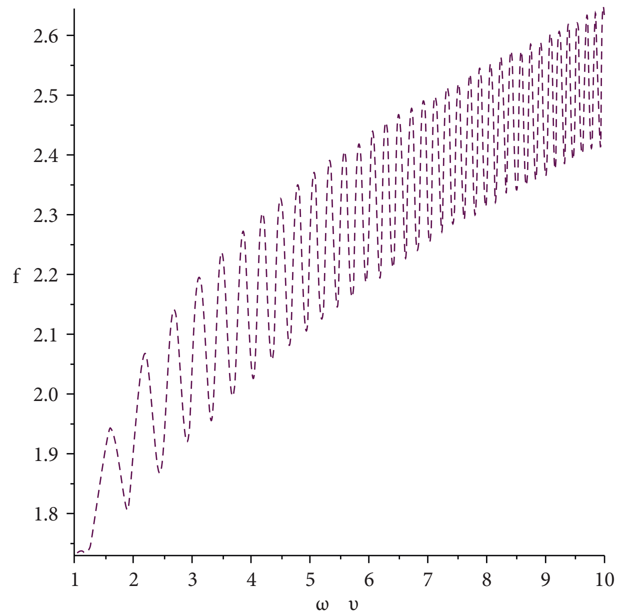

Figure 9: Exact solution of Example 2 for $(\varpi - \vartheta) \in [1, 10]$.

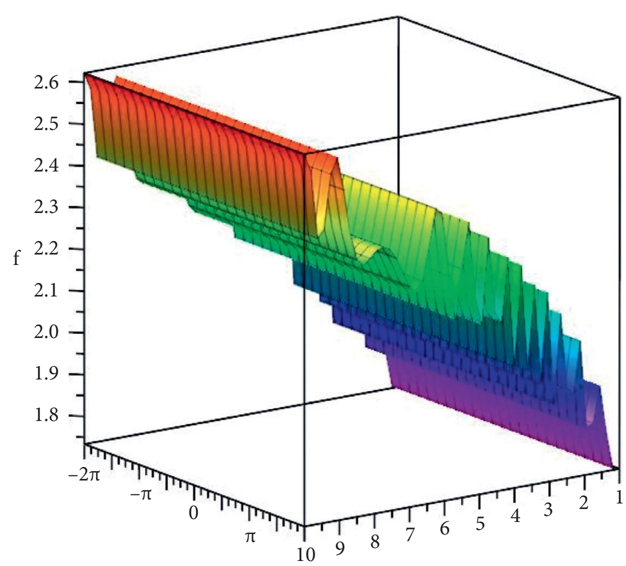

Figure 10: Exact solution of Example 2 for $(\varpi - \vartheta) \in [1, 10]$.

If $\mathfrak{h} \in C([1,2], \mathbb{R})$ be a differentiable function such that

$$\begin{aligned}
&\mathcal{N}\Bigg(\mathfrak{D}^{1}_{1/4}\mathfrak{h}(\varpi_1 - \vartheta_1) - \mathfrak{h}(\varpi_1 - \vartheta_1) - \frac{\sqrt{(\varpi_1 - \vartheta_1) + 1}}{6(\varpi_1 - \vartheta_1)^2 + 6}\sin(\mathfrak{h}(\varpi_1 - \vartheta_1)), \ldots, \\
&\mathfrak{D}^{1}_{1/4}\mathfrak{h}(\varpi_k - \vartheta_k) - \mathfrak{h}(\varpi_k - \vartheta_k) - \frac{\sqrt{(\varpi_k - \vartheta_k) + 1}}{6(\varpi_k - \vartheta_k)^2 + 6}\sin(\mathfrak{h}(\varpi_k - \vartheta_k)), \aleph \Bigg) \\
&\geqslant \mathbf{H}^{\mathbf{m},\mathbf{n}}_{\boldsymbol{\tau},\boldsymbol{\omega}}\left(-\frac{|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k|}{3\aleph} \right),
\end{aligned} \tag{82}$$

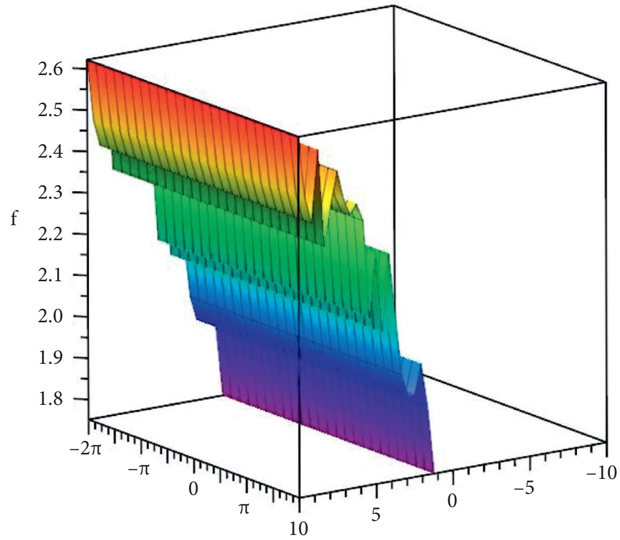

Figure 11: Exact solution of Example 2 for $(\varpi - \vartheta) \in [1, 10]$.

then $\mathfrak{h}$ is a solution of the inequality

$$\begin{aligned}
&\mathcal{N}\Bigg(\mathfrak{h}(\varpi_1 - \vartheta_1) - \mathfrak{h}_a \sum_{\varsigma=0}^{\infty} \frac{\left(4(\varpi_1 - \vartheta_1) - a\right)^{1/4})^{\varsigma}}{\Gamma(1+\tau\varsigma)} \\
&- \frac{1}{\Gamma(1/4)} \int_a^{(\varpi_1-\vartheta_1)} \sum_{\varsigma=0}^{\infty} \frac{\left(4(\varpi_1 - \vartheta_1) - a\right)^{1/4})^{\varsigma}}{\Gamma(1+\tau\varsigma)} \sum_{\varsigma=0}^{\infty} \frac{\left(\left(4(\lambdabar_1 - \lambdabar_1') - a\right)^{1/4}\right)^{-\varsigma}}{\Gamma(1+\tau\varsigma)} \\
&\frac{\sqrt{(\lambdabar_1 - \lambdabar_1') + 1}}{6(\lambdabar_1 - \lambdabar_1')^2 + 6} \sin(\mathfrak{h}(\lambdabar_1 - \lambdabar_1'))\left(\frac{(\lambdabar_1 - \lambdabar_1')}{t}\right)^s \left(\log \frac{t}{(\lambdabar_1 - \lambdabar_1')}\right)^{-3/4} \frac{d(\lambdabar_1 - \lambdabar_1')}{(\lambdabar_1 - \lambdabar_1')}, \ldots, \\
&\mathfrak{h}(\varpi_k - \vartheta_k) - \mathfrak{f}_a \sum_{\varsigma=0}^{\infty} \frac{\left(\left(4(\varpi_k - \vartheta_k) - a\right)^{1/4}\right)^{\varsigma}}{\Gamma(1+\tau\varsigma)} - \frac{1}{\Gamma(1/4)} \int_a^{(\varpi_k-\vartheta_k)} \sum_{\varsigma=0}^{\infty} \frac{\left(\left(4(\varpi_k - \vartheta_k) - a\right)^{1/4}\right)^{\varsigma}}{\Gamma(1+\tau\varsigma)} \\
&\sum_{\varsigma=0}^{\infty} \frac{\left(\left(4(\lambdabar_k - \lambdabar_k') - \mathfrak{c}\right)^{1/4}\right)^{-\varsigma}}{\Gamma(1+\tau\varsigma)} \frac{\sqrt{(\lambdabar_k - \lambdabar_k') + 1}}{6(\lambdabar_k - \lambdabar_k')^2 + 6} \sin(\mathfrak{h}(\lambdabar_k - \lambdabar_k')) \\
&\left(\frac{(\lambdabar_k - \lambdabar_k')}{t}\right)^s \left(\log \frac{t}{(\lambdabar_k - \lambdabar_k')}\right)^{-3/4} \frac{d(\lambdabar_k - \lambdabar_k')}{(\lambdabar_k - \lambdabar_k')}, \aleph\Bigg) \\
&\succcurlyeq \mathbf{H}_{\tau,\omega}^{\mathbf{m},\mathbf{n}} \Bigg| \left(\frac{1}{\Gamma(1/4)} \int_a^{(\varpi_1-\vartheta_1)} \sum_{\varsigma=0}^{\infty} \frac{\left(\left(4(\lambdabar_1 - \lambdabar_1') - a\right)^{1/4}\right)^{-\varsigma}}{\Gamma(1+\tau\varsigma)} \left(\frac{(\lambdabar_1 - \lambdabar_1')}{t}\right)^s \left(\log \frac{t}{(\lambdabar_1 - \lambdabar_1')}\right)^{-3/4} \frac{d(\lambdabar_1 - \lambdabar_1')}{(\lambdabar_1 - \lambdabar_1')}, \ldots, \right. \\
&\left. \frac{1}{\Gamma(1/4)} \int_a^{(\varpi_k-\vartheta_k)} \sum_{\varsigma=0}^{\infty} \frac{\left(\left(4(\lambdabar_k - \lambdabar_k') - a\right)^{1/4}\right)^{-\varsigma}}{\Gamma(1+\tau\varsigma)} \left(\frac{(\lambdabar_k - \lambdabar_k')}{t}\right)^s \left(\log \frac{t}{(\lambdabar_k - \lambdabar_k')}\right)^{-3/4} \frac{d(\lambdabar_k - \lambdabar_k')}{(\lambdabar_k - \lambdabar_k')} \Big| \frac{1}{\aleph/1/3 \sum_{\varsigma=0}^{\infty} (4)^{\varsigma}/\Gamma(1+\tau\varsigma)}\right),
\end{aligned} \tag{83}$$

and thus, we can find a unique differentiable function $\mathfrak{f} \in C([1,2], \mathbb{R})$ from (79) such that for each $(\varpi - \vartheta) \in [1,2]$, we have

$$\mathfrak{f}(\varpi_i - \vartheta_i) = \sum_{\varsigma=0}^{\infty} \frac{\left(4\left((\varpi_i - \vartheta_i) - 1\right)^{1/4}\right)^{\varsigma}}{\Gamma(1+\varsigma\tau)} + \frac{1}{\Gamma(1/4)} \int_1^{(\varpi_i - \vartheta_i)} \sum_{\varsigma=0}^{\infty} \frac{\left(4\left((\varpi_i - \vartheta_i) - 1\right)^{1/4}\right)^{\varsigma}}{\Gamma(1+\varsigma\tau)} \sum_{\varsigma=0}^{\infty} \frac{\left(2\left(\lambda_i - \lambda_i' - 1\right)^{1/4}\right)^{-\varsigma}}{\Gamma(1+\varsigma\tau)} \left(\frac{\lambda_i - \lambda_i'}{\mathrm{f}}\right)^{s} \left(\log \frac{\mathrm{f}}{\lambda_i - \lambda_i'}\right)^{-3/4} \cdot \frac{\sqrt{(\varpi_i - \vartheta_i) + 1}/6(\varpi_i - \vartheta_i)^2 + 6\sin\left(\mathfrak{h}(\varpi_i - \vartheta_i)\right)}{(\lambda_i - \lambda_i')} d(\lambda_i - \lambda_i'), \tag{84}$$

for any $i = 1, \ldots, k$. Therefore

$$d(\mathfrak{h}, \mathfrak{f}) \leq \frac{4}{1-\beth} \sum_{\varsigma=0}^{\infty} \frac{4^{\varsigma}}{\Gamma(1+\tau\varsigma)} \times \frac{1}{3},$$
$$\mathcal{N}\left(\mathfrak{h}(\varpi_1 - \vartheta_1) - \mathfrak{f}(\varpi_1 - \vartheta_1), \ldots, \mathfrak{h}(\varpi_k - \vartheta_k) - \mathfrak{f}(\varpi_k - \vartheta_k), \aleph\right)$$
$$\succcurlyeq \mathbf{H}_{\tau,\omega}^{\mathbf{m},\mathbf{n}}\left(-\frac{\left|\varpi_1 - \vartheta_1, \ldots, \varpi_k - \vartheta_k\right|}{\aleph(1-\beth)/4 \sum_{\varsigma=0}^{\infty} 2^{\varsigma}/\Gamma(1+\tau\varsigma) \times 1/3}\right). \tag{85}$$

Figures 9–11 show the graphs related to the exact solution of conformable FDE (79) for $\tau = 2, s = 1, t = 1/4$.

5. Conclusion

In this paper, we introduced the H-Fox function as a matrix value fuzzy control function, and by considering the matrix-valued fuzzy k-normed spaces, we investigated the stability of a class of conformable fractional differential equations with a constant coefficient. The alternative fixed-point theorem is used in different spaces. Therefore, we used the Radu–Mihet method, which is derived from the alternative fixed-point theorem, to investigate the existence of a unique solution and the Hyers–Ulam-H-Fox stability for the conformable fractional differential equations in the matrix-valued fuzzy k-normed spaces. The Riemann–Liouville fractional derivative and the Caputo fractional derivative have properties that cause high incompatibility and computational complexity in fractional calculations. To remove these obstacles and overcome these inconsistencies, an adaptable fractional derivative has been introduced, which we use because of these advantages.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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