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On Abelian Rings

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Abstract

Let α be an endomorphism of an arbitrary ring R with identity. In this note, we introduce the notion of α-abelian rings which generalizes abelian rings. We prove that α-reduced rings, α-symmetric rings, α-semicommutative rings and α-Armendariz rings are α-abelian. For a right principally projective ring R, we also prove that R is α-reduced if and only if R is α-symmetric if and only if R is α-semicommutative if and only if R is α-Armendariz if and only if R is α-Armendariz of power series type if and only if R is α-abelian.

Key word and phrases: α-reduced rings, α-symmetric rings, α-semicommutative rings, α-Armendariz rings, α-abelian rings.

1. Introduction

Throughout this paper R denotes an associative ring with identity 1 and α denotes a non-zero and non-identity endomorphism of a given ring with $\alpha(1) = 1$, and $\mathbf{1}$ denotes identity endomorphism, unless specified otherwise.

We write $R[x], R[[x]], R[x, x^{-1}]$ and $R[[x, x^{-1}]]$ for the polynomial ring, the power series ring, the Laurent polynomial ring and the Laurent power series ring over R, respectively. Consider

$$R[x, \alpha] = \left\{ \sum_{i=0}^{s} a_i x^i \ : \ s \geq 0, a_i \in R \right\},$$

$$R[[x, \alpha]] = \left\{ \sum_{i=0}^{\infty} a_i x^i \ : a_i \in R \right\},$$

$$R[x, x^{-1}, \alpha] = \left\{ \sum_{i=-s}^{t} a_i x^i \ : \ s \geq 0, t \geq 0, a_i \in R \right\},$$

$$R[[x, x^{-1}, \alpha]] = \left\{ \sum_{i=-s}^{\infty} a_i x^i \ : \ s \geq 0, a_i \in R \right\}.$$

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Each of these is an abelian group under an obvious addition operation. Moreover, $R[x,\alpha]$ becomes a ring under the following product operation:

For $f(x)=\sum_{i=0}^{s}a_ix^i, g(x)=\sum_{i=0}^{t}b_ix^i \in R[x,\alpha]$

$$f(x)g(x)=\sum_{k=0}^{s+t}\left(\sum_{i+j=k}a_i\alpha^i(b_j)\right)x^k.$$

Similarly, $R[[x,\alpha]]$ is a ring. The rings $R[x,\alpha]$ and $R[[x,\alpha]]$ are called the *skew polynomial extension* and the *skew power series extension of* R, respectively. If $\alpha\in Aut(R)$, then with a similar scalar product, $R[[x,x^{-1},\alpha]]$ (resp. $R[x,x^{-1},\alpha]$) becomes a ring. The rings $R[x,x^{-1},\alpha]$ and $R[[x,x^{-1},\alpha]]$ are called the *skew Laurent polynomial extension* and the *skew Laurent power series extension of* R, respectively.

In [8], *Baer rings* are introduced as rings in which the right(left) annihilator of every nonempty subset is generated by an idempotent. According to Clark [4], a ring R is said to be *quasi-Baer ring* if the right annihilator of each right ideal of R is generated(as a right ideal) by an idempotent. These definitions are left-right symmetric. A ring R is called *right principally quasi-Baer ring (or simply, right p.q.-Baer ring)* if the right annihilator of a principally right ideal of R is generated by an idempotent. Finally, a ring R is called *right principally projective ring (or simply, right p.p.-ring)* if the right annihilator of an element of R is generated by an idempotent [2].

2. Abelian Rings

In this section the notion of an α-abelian ring is introduced as a generalization of an abelian ring. We show that many results of abelian rings can be extended to α-abelian rings for this general settings.

The ring R is called *abelian* if every idempotent is central, that is, $ae=ea$ for any $e^2=e,\ a\in R$.

Definition 2.1 *A ring R is called α-abelian if, for any $a,b\in R$ and any idempotent $e\in R$,*
(i) $ea=ae$,
(ii) $ab=0$ if and only if $a\alpha(b)=0$.

So a ring R is *abelian* if and only if it is **1**-abelian.

Example 2.2 Let $\mathbb{Z}_4$ be the ring of integers modulo 4. Consider the ring $R=\{\begin{pmatrix} a & b\\ 0 & a\end{pmatrix} \mid a,b\in\mathbb{Z}_4\}$ with the usual matrix operations. Let $\alpha: R\to R$ be defined by $\alpha(\begin{pmatrix} a & b\\ 0 & a\end{pmatrix})=\begin{pmatrix} a & -b\\ 0 & a\end{pmatrix}$. It is easy to check that α is a homomorphism of R. We show that R is an α-abelian ring. Since R is commutative, R is abelian. To complete the proof we check that for any $r,s\in R$, $rs=0$ if and only if $r\alpha(s)=0$. We prove one way implication. The other way is similar. So let $r=\begin{pmatrix} a & b\\ 0 & a\end{pmatrix}$, $s=\begin{pmatrix} x & y\\ 0 & x\end{pmatrix}\in R$. Assume that $rs=0$ and

r and s are nonzero. Then we have $ax = 0$ and $ay + bx = 0$. If $a = 0$, then easy calculation shows that $r\alpha(s) = 0$. So we suppose $a \neq 0$. If $x = 0$ then $r\alpha(s) = 0$. Assume $x \neq 0$. Then $a = 2$ and $x = 2$. It implies $r\alpha(s) = 0$. Therefore R is α-abelian.

Lemma 2.3 *Let R be a ring such that for any $a, b \in R$, $ab = 0$ implies $a\alpha(b) = 0$, then $\alpha(e) = e$ for every idempotent $e \in R$.*

Proof. Since $e(1-e) = 0$ and $\alpha(1) = 1$, then $0 = e\alpha(1-e) = e - e\alpha(e)$. So $e = e\alpha(e)$. Further, $(1-e)e = 0$. Then $(1-e)\alpha(e) = 0$. Therefore, $\alpha(e) = e\alpha(e)$. So, we have $e = e\alpha(e)$ and $\alpha(e) = e\alpha(e)$. Hence, $e = \alpha(e)$. □

Example 2.4 shows that there exists an abelian ring, but it is not α-abelian.

Example 2.4 Let R be the ring $\mathbb{Z} \oplus \mathbb{Z}$ with the usual componentwise operations. It is clear that R is an abelian ring. Let $\alpha : R \to R$ be defined by $\alpha(a, b) = (b, a)$. Then $(1, 0)(0, 1) = 0$, but $(1, 0)\alpha(0, 1) \neq 0$. Hence R is not α-abelian.

The ring R is called *semicommutative* if $ab = 0$ implies $aRb = 0$, for any $a, b \in R$. A ring R is called α-*semicommutative* if $ab = 0$ implies $aR\alpha(b) = 0$, for any $a, b \in R$. Agayev and Harmanci studied basic properties of α-semicommutative rings and focused on the semicommutativity of subrings of matrix rings (see [1]). In this note, the ring R is said to be α-*semicommutative* if, for any $a, b \in R$,
(i) $ab = 0$ implies $aRb = 0$,
(ii) $ab = 0$ if and only if $a\alpha(b) = 0$.

It is clear that a ring R is semicommutative if and only if it is $\mathbf{1}$-semicommutative. The first part of Lemma 2.5 is proved in [7]. We give the proof for the sake of completeness.

Lemma 2.5 *If the ring R is α-semicommutative, then R is α-abelian. The converse holds if R is a right p.p.-ring.*

Proof. If e is an idempotent in R, then $e(1-e) = 0$. Since R is α-semicommutative, we have $ea(1-e) = 0$ for any $a \in R$ and so $ea = eae$. On the other hand, $(1-e)e = 0$ implies that $(1-e)ae = 0$, so we have $ae = eae$. Therefore, $ae = ea$. Suppose now R is an α-abelian and right p.p.-ring. Let $a, b \in R$ with $ab = 0$. Then $a \in r(b) = eR$ for some $e^2 = e \in R$ and so $be = 0$ and $a = ea$. Since R is α-abelian, we have $arb = earb = arbe = 0$ for any $r \in R$, that is, $aRb = 0$. Therefore R is α-semicommutative. □

Corollary 2.6 *If the ring R is semicommutative, then R is abelian. The converse holds if R is a right p.p.-ring.*

Corollary 2.7 *Let R be an α-abelian and right p.p-ring. Then $r(a) = r(aR)$, for any $a \in R$.*

Corollary 2.8 *Let R be an α-abelian and right p.p-ring. Then R is a right p.q.-Baer ring.*

Proof. It follows from Corollary 2.7. □

For a right R-module M, consider $M[x,\alpha] = \left\{\sum_{i=0}^{s} m_i x^i \ : \ s \geq 0, m_i \in M\right\}$. $M[x,\alpha]$ is an abelian group under an obvious addition operation and becomes a right module over $R[x;\alpha]$ under the following scalar product operation:

For $m(x) = \displaystyle\sum_{i=0}^{s} m_i x^i \ \in M[x,\alpha]$ and $f(x) = \displaystyle\sum_{i=0}^{t} a_i x^i \ \in R[x,\alpha]$

$$m(x)f(x) = \sum_{k=0}^{s+t}\left(\sum_{i+j=k} m_i \alpha^i(a_j)\right)x^k.$$

In [12], the ring R is called *Armendariz* if for any $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{s} b_j x^j \in R[x]$, $f(x)g(x) = 0$ implies $a_i b_j = 0$ for all i and j. This definition of Armendariz ring is extended to modules in [11]. A module M is called *α-Armendariz* if the following conditions (1) and (2) are satisfied, and the module M is called *α-Armendariz of power series type* if the following conditions (1) and (3) are satisfied:

(1) For $m \in M$ and $a \in R$, $ma = 0$ if and only if $m\alpha(a) = 0$.

(2) For any $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x,\alpha]$, $f(x) = \sum_{j=0}^{s} a_j x^j \in R[x,\alpha]$, $m(x)f(x) = 0$ implies $m_i\alpha^i(a_j) = 0$ for all i and j.

(3) For any $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x,\alpha]]$, $f(x) = \sum_{j=0}^{\infty} a_j x^j \in R[[x,\alpha]]$, $m(x)f(x) = 0$ implies $m_i\alpha^i(a_j) = 0$ for all i and j.

In this note, the ring R is called *α-Armendariz* (*α-Armendariz of power series type*) if R_R is *α-Armendariz* (*α-Armendariz of power series type*) module. Hence R is an Armendariz (Armendariz of power series type) ring if and only if R_R is an **1-**Armendariz (**1-**Armendariz of power series type) module.

Theorem 2.9 *If the ring R is α-Armendariz, then R is α-abelian. The converse holds if R is a right p.p.-ring.*

Proof. Let $f_1(x) = e - ea(1-e)x$, $f_2(x) = (1-e) - (1-e)aex$, $g_1(x) = 1 - e + ea(1-e)x$, $g_2(x) = e + (1-e)aex \in R[x,\alpha]$, where e is an idempotent in R and $a \in R$. Then $f_1(x)g_1(x) = 0$ and $f_2(x)g_2(x) = 0$. Since R is α-Armendariz, we have $ea(1-e)\alpha(1-e) = 0$. By Lemma 2.3, $\alpha(1-e) = 1-e$ and so $ea(1-e) = 0$. Similarly, $f_2(x)g_2(x) = 0$ implies that $(1-e)ae = 0$. Then $ae = eae = ea$, so R is α-abelian.

Suppose now R is an α-abelian and right p.p.-ring. Then R is abelian, and so every idempotent is central. By Lemma 2.3, $\alpha(e) = e$ for every idempotent $e \in R$. From Lemma 2.5, R is α-semicommutative, i.e., $ab = 0$ implies $aRb = 0$ for any $a, b \in R$. Let $f(x) = \displaystyle\sum_{i=0}^{s} a_i x^i$, $g(x) = \displaystyle\sum_{j=0}^{t} b_j x^j \ \in R[x,\alpha]$. Assume $f(x)g(x) = 0$. Then we have:

$$a_0 b_0 = 0 \qquad (1)$$

$$a_0 b_1 + a_1\alpha(b_0) = 0 \qquad (2)$$

$$a_0 b_2 + a_1\alpha(b_1) + a_2\alpha^2(b_0) = 0 \qquad (3)$$

$$\ldots$$

By hypothesis there exist idempotents $e_i \in R$ such that $r(a_i) = e_i R$ for all i. So $b_0 = e_0 b_0$ and $a_0 e_0 = 0$. Multiply (2) from the right by e_0, we have $0 = a_0 b_1 e_0 + a_1 \alpha(b_0) e_0 = a_0 e_0 b_1 + a_1 \alpha(b_0)\alpha(e_0) = a_1\alpha(b_0)$. By (2) $a_0 b_1 = 0$ and so $b_1 = e_0 b_1$. Again, multiply (3) from the right by e_0, we have $0 = a_0 b_2 e_0 + a_1 \alpha(b_1) e_0 + a_2 \alpha^2(b_0) e_0 = a_1\alpha(b_1) + a_2\alpha^2(b_0)$. Multiply this equation from right by e_1, we have $0 = a_1\alpha(b_1)e_1 + a_2\alpha^2(b_0)e_1 = a_2\alpha^2(b_0)$. Continuing in this way, we may conclude that $a_i\alpha^i(b_j) = 0$ for all $1 \leq i \leq s$ and $1 \leq j \leq t$. Hence R is α-Armendariz. This completes the proof. □

Corollary 2.10 *If the ring R is Armendariz, then R is abelian. The converse holds if R is a right p.p.-ring.*

Proposition 2.11 *If the ring R is α-Armendariz of power series type, then R is α-abelian. The converse holds if R is a right p.p.-ring.*

Proof. Similar to the proof of Theorem 2.9. □

Recall that a ring is *reduced* if it has no nonzero nilpotent elements. In [11], Lee and Zhou introduced α*-reduced* module. A module M is called α*-reduced* if, for any $m \in M$ and any $a \in R$,
(1) $ma = 0$ implies $mR \cap Ma = 0$
(2) $ma = 0$ if and only if $m\alpha(a) = 0$.

In this work, we call the ring R α*-reduced* if R_R is an α-reduced module. Hence R is a reduced ring if and only if R_R is an **1**-reduced module.

In [5], Hong et al. studied α-rigid rings. For an endomorphism α of a ring R, R is called α*-rigid* if $a\alpha(a) = 0$ implies $a = 0$ for any a in R. The relationship between α-rigid rings and α-skew Armendariz rings was studied in [6]. In fact, R is an α-Armendariz ring if and only if (1) R is an α-skew Armendariz ring and (2) $ab = 0$ if and only if $a\alpha(b) = 0$ for any a, b in R. Note that α-reduced ring is α-rigid. Really, let R be an α-reduced ring and $a\alpha(a) = 0$ for some a in R. Then $a^2 = 0$. Since R is reduced, we have $a = 0$. Further, by [5, Proposition 6], any α-reduced ring R is α-Armendariz. By Theorem 2.9, R is α-abelian. So, the first statement of Lemma 2.12 is a direct corollary of [5, Proposition 6].

Lemma 2.12 *If R is an α-reduced ring, then R is α-abelian. The converse holds if R is a right p.p.-ring.*

Proof. Let R be an α-abelian and right *p.p*-ring. Suppose $ab = 0$ for $a, b \in R$. If $x \in aR \cap Rb$, then there exist $r_1, r_2 \in R$ such that $x = ar_1 = r_2 b$. Since R is right *p.p*-ring, $ab = 0$ implies that $b \in r(a) = eR$ for some idempotent $e^2 = e \in R$. Then $b = eb$ and $xe = ar_1 e = r_2 be$. Since R is α-abelian and $ae = 0$, we have $ar_1 e = aer_1 = r_2 be = r_2 eb = r_2 b = 0$. Hence $aR \cap Rb = 0$, that is, R is α-reduced. □

Corollary 2.13 *If R is a reduced ring, then R is abelian. The converse holds if R is a right p.p.-ring.*

According to Lambek [10], a ring R is called *symmetric* if whenever $a, b, c \in R$ satisfy $abc = 0$, we have $bac = 0$; it is easily seen that this is a left-right symmetric concept. We now introduce α-symmetric rings as a generalization of symmetric rings.

Definition 2.14 *The ring R is called α-symmetric if, for any $a, b, c \in R$,*
(i) $abc = 0$ implies $acb = 0$,
(ii) $ab = 0$ if and only if $a\alpha(b) = 0$.

It is clear that a ring R is symmetric if and only if it is $\mathbf{1}$-symmetric.

Theorem 2.15 *Let R be a right p.p-ring. Then the following are equivalent:*
(1) R is α-reduced.
(2) R is α-symmetric.
(3) R is α-semicommutative.
(4) R is α-Armendariz.
(5) R is α-Armendariz of power series type.
(6) R is α-abelian.

Proof. (1)$\Leftrightarrow$ (6) From Lemma 2.12.
(4)$\Leftrightarrow$ (6) Clear from Theorem 2.9.
(3)$\Leftrightarrow$ (6) From Lemma 2.5.
(5)$\Leftrightarrow$ (6) From Proposition 2.11.
(2)$\Rightarrow$ (3) Let $a, b \in R$ with $ab = 0$. By hypothesis, $abc = 0$ implies $acb = 0$ for all $c \in R$. Hence $aRb = 0$ and so R is α-semicommutative.
(3)$\Rightarrow$ (2) Assume that $abc = 0$, for any $a, b, c \in R$. Since R is right p.p.-ring, $c \in r(ab) = eR$ for some idempotent $e \in R$. Then $c = ec$ and $abe = 0$, so $acbe = 0$. We have already proved that semicommutativity implied being abelian, then $acbe = aecb$. Now $acb = aecb = acbe = 0$. It completes the proof. $\square$

Corollary 2.16 *Let R be a Baer ring. Then the following are equivalent:*
(1) R is α-reduced.
(2) R is α-symmetric.
(3) R is α-semicommutative.
(4) R is α-Armendariz.
(5) R is α-Armendariz of power series type.
(6) R is α-abelian.

One may suspect that if R is an abelian ring, then $R[x, \alpha]$ is abelian also. But this is not the case.

Example 2.17 Let F be any field, $R = \left\{ \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & 0 & u \end{pmatrix} \mid a, b, u, v \in F \right\}$ and $\alpha : R \to R$ be defined by

$$\alpha \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & 0 & u \end{pmatrix} = \begin{pmatrix} u & v & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}, \text{ where } \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & 0 & u \end{pmatrix} \in R$$

Since R is commutative, R is abelian. We claim that $R[x,\alpha]$ is not an abelian ring. Let e_{ij} denote the 4×4 matrix units having alone 1 as its (i,j)-entry and all other entries 0. Consider $e = e_{11}+e_{22}$ and $f = e_{33}+e_{44} \in R$ and $e(x) = e + fx \in R[x;\alpha]$. Then $e(x)^2 = e(x)$, $ef = fe = 0$, $e^2 = e$, $f^2 = f$, $\alpha(e) = f$, $\alpha(f) = e$. An easy calculation reveals that $e(x)e_{12} = e_{12} + e_{34}x$, but $e_{12}e(x) = e_{12}$. Hence $R[x,\alpha]$ is not an abelian ring.

Lemma 2.18 *If R is an α-abelian ring, then the idempotents of $R[x,\alpha]$ belong to R, therefore $R[x,\alpha]$ is an abelian ring.*

Proof. Let R be α-abelian and $e(x) = \sum_{i=0}^{t} e_i x^i$ be an idempotent in $R[x,\alpha]$. Since $e(x)^2 = e(x)$, we have

$$e_0^2 = e_0 \qquad (1)$$

$$e_0e_1 + e_1\alpha(e_0) = e_1 \qquad (2)$$

$$e_0e_2 + e_1\alpha(e_1) + e_2\alpha^2(e_0) = e_2 \qquad (3)$$

$$...$$

Since R is α-abelian, R is abelian, and so every idempotent is central. By Lemma 2.3, $\alpha(e) = e$ for every idempotent $e \in R$. Then (2) becomes $e_0e_1 + e_1e_0 = e_1$ and so $e_1 = 0$. Since e_0 is central idempotent, (3) becomes $e_0e_2 + e_2e_0 = e_2$ and so $e_2 = 0$. Similarly, it can be shown that $e_i = 0$ for $i = 1, 2, ..., t$. This completes the proof. □

Lemma 2.19 *If $R[x,\alpha]$ is an abelian ring, then $\alpha(e) = e$ for every idempotent $e \in R$.*

Proof. Since $R[x,\alpha]$ is abelian, we have $f(x)e(x) = e(x)f(x)$ for any $f(x), e(x)^2 = e(x) \in R[x,\alpha]$. In particular, $xe = ex$ for every idempotent $e \in R$. Hence $xe = ex = \alpha(e)x$ and so $\alpha(e) = e$. □

Lemma 2.20 *If $R[x,\alpha]$ is an abelian ring, then the idempotents of $R[x,\alpha]$ belong to R.*

Proof. Similar to the proof of Lemma 2.18. □

Theorem 2.21 *If R is an α-abelian ring, then $R[x,\alpha]$ is abelian. The converse holds if $R[x,\alpha]$ is a right p.p.-ring.*

Proof. If R is α-abelian, by Lemma 2.18, $R[x,\alpha]$ is abelian. Suppose that $R[x,\alpha]$ be an abelian and right p.p.-ring. It is clear that $ae = ea$ for any $a, e^2 = e \in R$. Suppose $ab = 0$ for any $a, b \in R$. Since R is right p.p.-ring, we have $b \in r(a) = eR$, $b = eb$. So $a\alpha(b) = a\alpha(eb) = ae\alpha(b) = 0$. Conversely, let $a\alpha(b) = 0$. Then $axb = 0$. Since $R[x,\alpha]$ is right p.p.-ring, we have $b \in r_{R[x,\alpha]}(ax) = eR[x,\alpha]$ for some idempotent $e \in R[x,\alpha]$. So $b = eb$, $axe = 0$. By Lemma 2.20, $e \in R$. Hence $ae = 0$ and $ab = aeb = 0$. Therefore R is α-abelian. □

Lemma 2.22 *Let R be an α-abelian ring. If for any countable subset X of R, $r(X) = eR$, where $e^2 = e \in R$, then*
(1) $R[[x, \alpha]]$ is a right p.p.-ring.
(2) If α is an automorphism of R, then $R[[x, x^{-1}, \alpha]]$ is a right p.p.-ring.

Proof. Let $a \in R$. Since $\{a\}$ is countable subset of R, $r(a) = eR$, i.e., R is a right p.p.-ring. Then from Theorem 2.15, R is α-Armendariz of power series type. By [11, Theorem 2.11.(1)(c), Theorem 2.11.(2)(c)], $R[[x, \alpha]]$ and $R[[x, x^{-1}, \alpha]]$ are right p.p.-rings. □

Theorem 2.23 *Let R be an α-abelian ring. Then we have:*
(1) R is a right p.p.-ring if and only if $R[x, \alpha]$ is a right p.p.-ring.
(2) R is a Baer ring if and only if $R[x, \alpha]$ is a Baer ring.
(3) R is a right p.q.-Baer ring if and only if $R[x, \alpha]$ is a right p.q.-Baer ring.
(4) R is a Baer ring if and only if $R[[x, \alpha]]$ is a Baer ring.
Let $\alpha \in Aut(R)$.
(5) R is a Baer ring if and only if $R[x, x^{-1}, \alpha]$ is a Baer ring.
(6) R is a right p.p.-ring if and only if $R[x, x^{-1}, \alpha]$ is a right p.p.-ring.
(7) R is a Baer ring if and only if $R[[x, x^{-1}, \alpha]]$ is a Baer ring.

Proof. (1) "$\Rightarrow$": Let $f(x) = a_0 + a_1x + ... + a_tx^t \in R[x, \alpha]$. We claim that $r_{R[x,\alpha]}(f(x)) = eR[x, \alpha]$, where $e = e_0e_1...e_t$, $e_i^2 = e_i$ and $r_R(a_i) = e_iR$, $i = 0, 1, ..., t$. By hypothesis and Lemma 2.3, $f(x)e = a_0e_0e_1...e_t + a_1e_1e_0e_2...e_tx + ... + a_te_te_0e_1...e_{t-1}x^t = 0$. Then $eR[x] \subseteq r_{R[x,\alpha]}(f(x))$. Let $g(x) = b_0 + b_1x + ... + b_nx^n \in r_{R[x,\alpha]}(f(x))$. Then $f(x)g(x) = 0$. Since R is an abelian and right p.p.-ring, by Theorem 2.9, R is Armendariz. So $a_ib_j = 0$ and this implies $b_j \in r_R(a_i) = e_iR$, and then $b_j = e_ib_j$ for any i. Therefore $g(x) = eg(x) \in eR[x, \alpha]$. This completes the proof of (1) "$\Rightarrow$".

" $\Leftarrow$ ": Let $a \in R$. Then there exists $e(x)^2 = e(x) \in R[x, \alpha]$ such that $r_{R[x,\alpha]}(a) = e(x)R[x, \alpha]$. Then the constant term, e_0 say, of $e(x)$ is non-zero, and e_0 is an idempotent in R. So $e_0R \subset r_R(a)$. Now let $b \in r_R(a)$. Since $r_R(a) \subset r_{R[x,\alpha]}(a)$, $ab = 0$ implies that $b = e(x)b$ and so $b = e_0b$. Hence $r_R(a) \subset e_0R$, that is, $r_R(a) = e_0R$. Therefore R is a right p.p.-ring.

(2) " $\Rightarrow$ ": Since R is Baer, R is a right p.p.-ring. By Lemma 2.5, R is Armendariz. Then from [11, Theorem 2.5.1(a)], $R[x, \alpha]$ is Baer.

" $\Leftarrow$ ": Let $R[x, \alpha]$ be a Baer ring and X be a subset of R. There exists $e(x)^2 = e(x) = e_0 + e_1x + ... + e_nx^n \in R[x, \alpha]$ such that $r_{R[x;\alpha]}(X) = e(x)R[x, \alpha]$. We claim that $r_R(X) = e_0R$. If $a \in r_R(X)$, then $a = e(x)a$ and so $a = e_0a$. Hence $r_R(X) \subset e_0R$. Since $Xe(x) = 0$, we have $Xe_0 = 0$, that is, $e_0R \subset r_R(X)$. Then R is a Baer ring.

(3) " $\Rightarrow$ ": Let $f(x) = a_0 + a_1x + ... + a_tx^t \in R[x, \alpha]$. We prove $r_{R[x,\alpha]}(f(x)R[x, \alpha]) = e(x)R[x, \alpha]$, where $e(x) = e_0e_1...e_t$, $r_R(a_iR) = e_iR$. Since R is abelian, for any $h(x) \in R[x, \alpha]$ $f(x)h(x)e(x) = 0$. Then $e(x)R[x, \alpha] \subset r_{R[x,\alpha]}(f(x)R[x, \alpha])$. Let $g(x) = b_0 + b_1x + ... + b_nx^n \in r_{R[x,\alpha]}(f(x)R[x, \alpha])$. Then $f(x)R[x, \alpha]g(x) = 0$ and so, $f(x)Rg(x) = 0$. From last equality we have $a_0Rb_0 = 0$. Hence $b_0 \in r_R(a_0R) = e_0R$. It follows that $b_0 = e_0b_0$. Also for any $r \in R$, the coefficient of x is equal to $a_0rb_1 + a_1\alpha(rb_0)$.

Hence $a_0rb_1 + a_1\alpha(rb_0) = 0$. Multiplying the equation $a_0rb_1 + a_1\alpha(rb_0) = 0$ from the right by e_0, we have $a_1\alpha(rb_0e_0) = 0$, that is, $a_1\alpha(rb_0) = 0$. Since R is α-abelian, $a_1rb_0 = 0$. This implies $a_1Rb_0 = 0$. Then $b_0 \in r_R(a_1R) = e_1R$ and $b_1 \in r_R(a_0R) = e_0R$. So, $b_0 = e_1b_0$ and $b_1 = e_0b_1$. Again for any $r \in R$, $a_0rb_2 + a_1rb_1 + a_2rb_0 = 0$. Multiplying this equality from right by e_0e_1 and using previous results, we have $a_2rb_0 = 0$. Then $b_0 \in r_R(a_2R) = e_2R$. So $b_0 = e_2b_0$. Continuing this process we have $b_i = e_jb_i$ for any i, j. This implies $g(x) = e_0e_1...e_tg(x)$. So, $R[x,\alpha]$ is a right p.q.-Baer ring.
" $\Leftarrow$ ": Let $a \in R$. Then $r_{R[x,\alpha]}(aR[x,\alpha]) = e(x)R[x,\alpha]$, where $e(x)^2 = e(x) \in R[x,\alpha]$. By Lemma 2.18, $e(x) = e_0 \in R$. Since $aR[x,\alpha]e(x) = 0$, $aR[x,\alpha]e_0 = 0$ and $aRe_0 = 0$. So, $e_0R \subset r_R(aR)$. Let $r \in r_R(aR) = r_R(aR[x,\alpha]) \subset r_{R[x,\alpha]}(aR[x,\alpha]) = e(x)R[x,\alpha]$. Then $e(x)r = r$. This implies $e_0r = r$ and so $r \in e_0R$. Therefore $r_R(aR[x,\alpha]) = e_0R$, i.e., R is a right p.q.-Baer ring.
(4) By Corollary 2.16, every abelian and Baer ring is Armendariz of power series type, so the proof follows from [11, Theorem 2.5 (1)(b)].
(5) By Corollary 2.16, R is α-Armendariz, then proof follows from [11, Theorem 2.5 (2)(a)].
(6) Since every α-abelian and right p.p.-ring is α-Armendariz by Theorem 2.9, the proof follows from [11, Theorem 2.11 (2)(a)].
(7) By Corollary 2.16, every abelian and Baer ring is Armendariz of power series type, it follows from [11, Theorem 2.5 (2)(b)]. □

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