On Abelian Rings

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Abstract

Let $\alpha$ be an endomorphism of an arbitrary ring $R$ with identity. In this note, we introduce the notion of $\alpha$-abelian rings which generalizes abelian rings. We prove that $\alpha$-reduced rings, $\alpha$-symmetric rings, $\alpha$-semicommutative rings and $\alpha$-Armendariz rings are $\alpha$-abelian. For a right principally projective ring $R$, we also prove that $R$ is $\alpha$-reduced if and only if $R$ is $\alpha$-symmetric if and only if $R$ is $\alpha$-semicommutative if and only if $R$ is $\alpha$-Armendariz if and only if $R$ is $\alpha$-Armendariz of power series type if and only if $R$ is $\alpha$-abelian.

Key word and phrases: $\alpha$-reduced rings, $\alpha$-symmetric rings, $\alpha$-semicommutative rings, $\alpha$-Armendariz rings, $\alpha$-abelian rings.

1. Introduction

Throughout this paper $R$ denotes an associative ring with identity $1$ and $\alpha$ denotes a non-zero and non-identity endomorphism of a given ring with $\alpha(1) = 1$, and $1$ denotes identity endomorphism, unless specified otherwise.

We write $R[x], R[[x]], R[x, x^{-1}]$ and $R[[x, x^{-1}]]$ for the polynomial ring, the power series ring, the Laurent polynomial ring and the Laurent power series ring over $R$, respectively. Consider

\[
R[x, \alpha] = \left\{ \sum_{i=0}^{s} a_i x^i : s \geq 0, a_i \in R \right\},
\]

\[
R[[x, \alpha]] = \left\{ \sum_{i=0}^{\infty} a_i x^i : a_i \in R \right\},
\]

\[
R[x, x^{-1}, \alpha] = \left\{ \sum_{i=-s}^{t} a_i x^i : s \geq 0, t \geq 0, a_i \in R \right\},
\]

\[
R[[x, x^{-1}], \alpha]] = \left\{ \sum_{i=-s}^{\infty} a_i x^i : s \geq 0, a_i \in R \right\}.
\]
Each of these is an abelian group under an obvious addition operation. Moreover, \( R[x, \alpha] \) becomes a ring under the following product operation:

\[
\text{For } f(x) = \sum_{i=0}^{s} a_i x^i, g(x) = \sum_{i=0}^{t} b_i x^i \in R[x, \alpha]
\]

\[
f(x)g(x) = \sum_{k=0}^{s+t} \left( \sum_{i+j=k} a_i \alpha^i(b_j) \right) x^k.
\]

Similarly, \( R[[x, \alpha]] \) is a ring. The rings \( R[x, \alpha] \) and \( R[[x, \alpha]] \) are called the \textit{skew polynomial extension} and the \textit{skew power series extension} of \( R \), respectively. If \( \alpha \in \text{Aut}(R) \), then with a similar scalar product, \( R[[x, x^{-1}, \alpha]] \) (resp. \( R[x, x^{-1}, \alpha] \)) becomes a ring. The rings \( R[x, x^{-1}, \alpha] \) and \( R[[x, x^{-1}, \alpha]] \) are called the \textit{skew Laurent polynomial extension} and the \textit{skew Laurent power series extension} of \( R \), respectively.

In [8], \textit{Baer rings} are introduced as rings in which the right(left) annihilator of every nonempty subset is generated by an idempotent. According to Clark [4], a ring \( R \) is said to be \textit{quasi-Baer ring} if the right annihilator of each right ideal of \( R \) is generated (as a right ideal) by an idempotent. These definitions are left-right symmetric. A ring \( R \) is called \textit{right principally quasi-Baer ring} (or simply, right p.q.-Baer ring) if the right annihilator of a principally right ideal of \( R \) is generated by an idempotent. Finally, a ring \( R \) is called \textit{right principally projective ring} (or simply, right p.p.-ring) if the right annihilator of an element of \( R \) is generated by an idempotent [2].

2. Abelian Rings

In this section the notion of an \( \alpha \)-abelian ring is introduced as a generalization of an abelian ring. We show that many results of abelian rings can be extended to \( \alpha \)-abelian rings for this general settings.

The ring \( R \) is called \textit{abelian} if every idempotent is central, that is, \( ae = ea \) for any \( e^2 = e \), \( a \in R \).

**Definition 2.1** A ring \( R \) is called \( \alpha \)-abelian if, for any \( a, b \in R \) and any idempotent \( e \in R \),

(i) \( ea = ae \),

(ii) \( ab = 0 \) if and only if \( a\alpha(b) = 0 \).

So a ring \( R \) is \textit{abelian} if and only if it is \textbf{1}-abelian.

**Example 2.2** Let \( \mathbb{Z}_4 \) be the ring of integers modulo 4. Consider the ring \( R = \{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_4 \} \) with the usual matrix operations. Let \( \alpha : R \to R \) be defined by \( \alpha( \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} ) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix} \). It is easy to check that \( \alpha \) is a homomorphism of \( R \). We show that \( R \) is an \( \alpha \)-abelian ring. Since \( R \) is commutative, \( R \) is abelian. To complete the proof we check that for any \( r, s \in R \), \( rs = 0 \) if and only if \( r\alpha(s) = 0 \). We prove one way implication. The other way is similar. So let \( r = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \), \( s = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \in R \). Assume that \( rs = 0 \) and
r and s are nonzero. Then we have \( ax = 0 \) and \( ay + bx = 0 \). If \( a = 0 \), then easy calculation shows that \( r\alpha(s) = 0 \). So we suppose \( a \neq 0 \). If \( x = 0 \) then \( r\alpha(s) = 0 \). Assume \( x \neq 0 \). Then \( a = 2 \) and \( x = 2 \). It implies \( r\alpha(s) = 0 \). Therefore \( R \) is \( \alpha \)-abelian.

**Lemma 2.3** Let \( R \) be a ring such that for any \( a, b \in R \), \( ab = 0 \) implies \( a\alpha(b) = 0 \), then \( \alpha(e) = e \) for every idempotent \( e \in R \).

**Proof.** Since \( e(1-e) = 0 \) and \( \alpha(1) = 1 \), then \( 0 = e\alpha(1-e) = e-e\alpha(e) \). So \( e = e\alpha(e) \). Further, \( (1-e)e = 0 \). Then \( (1-e)\alpha(e) = 0 \). Therefore, \( \alpha(e) = e\alpha(e) \). So, we have \( e = e\alpha(e) \) and \( \alpha(e) = e\alpha(e) \). Hence, \( e = \alpha(e) \).

Example 2.4 shows that there exists an abelian ring, but it is not \( \alpha \)-abelian.

**Example 2.4** Let \( R \) be the ring \( \mathbb{Z} \oplus \mathbb{Z} \) with the usual componentwise operations. It is clear that \( R \) is an abelian ring. Let \( \alpha : R \to R \) be defined by \( \alpha(a, b) = (b, a) \). Then \( (1, 0)(0, 1) = 0 \), but \( (1, 0)\alpha(0, 1) \neq 0 \). Hence \( R \) is not \( \alpha \)-abelian.

The ring \( R \) is called \emph{semicommutative} if \( ab = 0 \) implies \( aRb = 0 \), for any \( a, b \in R \). A ring \( R \) is called \( \alpha \)-\emph{semicommutative} if \( ab = 0 \) implies \( a\alpha(b) = 0 \), for any \( a, b \in R \). Agayev and Harmanci studied basic properties of \( \alpha \)-semicommutative rings and focused on the semicommutativity of subrings of matrix rings (see [1]). In this note, the ring \( R \) is said to be \( \alpha \)-\emph{semicommutative} if, for any \( a, b \in R \),

(i) \( ab = 0 \) implies \( aRb = 0 \),

(ii) \( ab = 0 \) if and only if \( a\alpha(b) = 0 \).

It is clear that a ring \( R \) is semicommutative if and only if it is \emph{1}-semicommutative. The first part of Lemma 2.5 is proved in [7]. We give the proof for the sake of completeness.

**Lemma 2.5** If the ring \( R \) is \( \alpha \)-semicommutative, then \( R \) is \( \alpha \)-abelian. The converse holds if \( R \) is a right p.p.-ring.

**Proof.** If \( e \) is an idempotent in \( R \), then \( e(1-e) = 0 \). Since \( R \) is \( \alpha \)-semicommutative, we have \( e\alpha(1-e) = 0 \) for any \( a \in R \) and so \( ea = eae \). On the other hand, \( (1-e)e = 0 \) implies that \( (1-e)ae = 0 \), so we have \( ae = eae \). Therefore, \( ae = ea \). Suppose now \( R \) is an \( \alpha \)-abelian and right p.p.-ring. Let \( a, b \in R \) with \( ab = 0 \). Then \( a \in r(b) = eR \) for some \( e^2 = e \in R \) and so \( be = 0 \) and \( a = ea \). Since \( R \) is \( \alpha \)-abelian, we have \( arb = earb = arbe = 0 \) for any \( r \in R \), that is, \( aRb = 0 \). Therefore \( R \) is \( \alpha \)-semicommutative.

**Corollary 2.6** If the ring \( R \) is semicommutative, then \( R \) is abelian. The converse holds if \( R \) is a right p.p.-ring.

**Corollary 2.7** Let \( R \) be an \( \alpha \)-abelian and right p.p.-ring. Then \( r(a) = r(aR) \), for any \( a \in R \).

**Corollary 2.8** Let \( R \) be an \( \alpha \)-abelian and right p.p.-ring. Then \( R \) is a right p.q.-Baer ring.

**Proof.** It follows from Corollary 2.7.
For a right $R$-module $M$, consider $M[x, \alpha] = \{ \sum_{i=0}^{s} m_{i}x^{i} : s \geq 0, m_{i} \in M \}$. $M[x, \alpha]$ is an abelian group under an obvious addition operation and becomes a right module over $R[x, \alpha]$ under the following scalar product operation:

For $m(x) = \sum_{i=0}^{s} m_{i}x^{i} \in M[x, \alpha]$ and $f(x) = \sum_{i=0}^{t} a_{i}x^{i} \in R[x, \alpha]$

$$m(x)f(x) = \sum_{k=0}^{s+t} \left( \sum_{i+j=k} m_{i}a^{j}(a_{j}) \right) x^{k}.$$

In [12], the ring $R$ is called Armendariz if for any $f(x) = \sum_{i=0}^{s} a_{i}x^{i}, g(x) = \sum_{j=0}^{s} b_{j}x^{j} \in R[x]$, $f(x)g(x) = 0$ implies $a_{i}b_{j} = 0$ for all $i$ and $j$. This definition of Armendariz ring is extended to modules in [11]. A module $M$ is called $\alpha$-Armendariz if the following conditions (1) and (2) are satisfied, and the module $M$ is called $\alpha$-Armendariz of power series type if the following conditions (1) and (3) are satisfied:

(1) For $m \in M$ and $a \in R$, $ma = 0$ if and only if $m\alpha(a) = 0$.

(2) For any $m(x) = \sum_{i=0}^{s} m_{i}x^{i} \in M[x, \alpha]$, $f(x) = \sum_{j=0}^{s} a_{j}x^{j} \in R[x, \alpha]$, $m(x)f(x) = 0$ implies $m_{i}\alpha^{i}(a_{j}) = 0$ for all $i$ and $j$.

(3) For any $m(x) = \sum_{i=0}^{s} m_{i}x^{i} \in M[[x, \alpha]]$, $f(x) = \sum_{i=0}^{\infty} a_{j}x^{j} \in R[[x, \alpha]]$, $m(x)f(x) = 0$ implies $m_{i}\alpha^{i}(a_{j}) = 0$ for all $i$ and $j$.

In this note, the ring $R$ is called $\alpha$-Armendariz (\alpha-Armendariz of power series type) if $R_{R}$ is $\alpha$-Armendariz (\alpha-Armendariz of power series type) module. Hence $R$ is an Armendariz (Armendariz of power series type) ring if and only if $R_{R}$ is an $\textbf{1}$-Armendariz (\textbf{1}-Armendariz of power series type) module.

**Theorem 2.9** If the ring $R$ is $\alpha$-Armendariz, then $R$ is $\alpha$-abelian. The converse holds if $R$ is a right p.p.-ring.

**Proof.** Let $f_{1}(x) = e - ea(1 - e)x$, $f_{2}(x) = (1 - e) - (1 - e)ax$, $g_{1}(x) = 1 - e + ea(1 - e)x$, $g_{2}(x) = e + (1 - e)ax \in R[x, \alpha]$, where $e$ is an idempotent in $R$ and $a \in R$. Then $f_{1}(x)g_{1}(x) = 0$ and $f_{2}(x)g_{2}(x) = 0$. Since $R$ is $\alpha$-Armendariz, we have $ea(1 - e)\alpha(1 - e) = 0$. By Lemma 2.3, $\alpha(1 - e) = 1 - e$ and so $ea(1 - e) = 0$. Similarly, $f_{2}(x)g_{2}(x) = 0$ implies that $(1 - e)ae = 0$. Then $ae = eae = ea$, so $R$ is $\alpha$-abelian.

Suppose now $R$ is an $\alpha$-abelian and right p.p.-ring. Then $R$ is abelian, and so every idempotent is central. By Lemma 2.3, $\alpha(e) = e$ for every idempotent $e \in R$. From Lemma 2.5, $R$ is $\alpha$-semicommutative, i.e., $ab = 0$ implies $aRb = 0$ for any $a, b \in R$. Let $f(x) = \sum_{i=0}^{s} a_{i}x^{i}$, $g(x) = \sum_{j=0}^{t} b_{j}x^{j} \in R[x, \alpha]$. Assume $f(x)g(x) = 0$. Then we have:

1. $a_{0}b_{0} = 0$  
2. $a_{0}b_{1} + a_{1}\alpha(b_{0}) = 0$  
3. $a_{0}b_{2} + a_{1}\alpha(b_{1}) + a_{2}\alpha^{2}(b_{0}) = 0$  

...
Corollary 2.10 If the ring $R$ is Armendariz, then $R$ is abelian. The converse holds if $R$ is a right p.p.-ring.

Proposition 2.11 If the ring $R$ is $\alpha$-Armendariz of power series type, then $R$ is $\alpha$-abelian. The converse holds if $R$ is a right p.p.-ring.

Proof. Similar to the proof of Theorem 2.9.

Recall that a ring is reduced if it has no nonzero nilpotent elements. In [11], Lee and Zhou introduced $\alpha$-reduced module. A module $M$ is called $\alpha$-reduced if, for any $m \in M$ and any $a \in R$,

1. $ma = 0$ implies $mR \cap Ma = 0$
2. $ma = 0$ if and only if $ma(a) = 0$.

In this work, we call the ring $R$ $\alpha$-reduced if $R_R$ is an $\alpha$-reduced module. Hence $R$ is a reduced ring if and only if $R_R$ is an 1-reduced module.

In [5], Hong et al. studied $\alpha$-rigid rings. For an endomorphism $\alpha$ of a ring $R$, $R$ is called $\alpha$-rigid if $a\alpha(a) = 0$ implies $a = 0$ for any $a$ in $R$. The relationship between $\alpha$-rigid rings and $\alpha$-skew Armendariz rings was studied in [6]. In fact, $R$ is an $\alpha$-Armendariz ring if and only if (1) $R$ is an $\alpha$-skew Armendariz ring and (2) $ab = 0$ if and only if $a\alpha(b) = 0$ for any $a, b$ in $R$. Note that $\alpha$-reduced ring is $\alpha$-rigid. Really, let $R$ be an $\alpha$-reduced ring and $a\alpha(a) = 0$ for some $a$ in $R$. Then $a^2 = 0$. Since $R$ is reduced, we have $a = 0$. Further, by [5, Proposition 6], any $\alpha$-reduced ring $R$ is $\alpha$-Armendariz. By Theorem 2.9, $R$ is $\alpha$-abelian. So, the first statement of Lemma 2.12 is a direct corollary of [5, Proposition 6].

Lemma 2.12 If $R$ is an $\alpha$-reduced ring, then $R$ is $\alpha$-abelian. The converse holds if $R$ is a right p.p.-ring.

Proof. Let $R$ be an $\alpha$-abelian and right p.p.-ring. Suppose $ab = 0$ for $a, b \in R$. If $x \in aR \cap Rb$, then there exist $r_1, r_2 \in R$ such that $x = ar_1 = r_2b$. Since $R$ is right p.p.-ring, $ab = 0$ implies that $b \in r(a) = eR$ for some idempotent $e^2 = e \in R$. Then $b = eb$ and $xe = ar_1e = r_2be$. Since $R$ is $\alpha$-abelian and $ae = 0$, we have $ar_1e = aer_1 = r_2be = r_2be = r_2b = 0$. Hence $aR \cap Rb = 0$, that is, $R$ is $\alpha$-reduced.

Corollary 2.13 If $R$ is a reduced ring, then $R$ is abelian. The converse holds if $R$ is a right p.p.-ring.

According to Lambek [10], a ring $R$ is called symmetric if whenever $a, b, c \in R$ satisfy $abc = 0$, we have $bac = 0$; it is easily seen that this is a left-right symmetric concept. We now introduce $\alpha$-symmetric rings as a generalization of symmetric rings.
Definition 2.14 The ring $R$ is called $\alpha$-symmetric if, for any $a, b, c \in R$,
(i) $abc = 0$ implies $acb = 0$,
(ii) $ab = 0$ if and only if $a\alpha(b) = 0$.

It is clear that a ring $R$ is symmetric if and only if it is $1$-symmetric.

Theorem 2.15 Let $R$ be a right p.p-ring. Then the following are equivalent:
(1) $R$ is $\alpha$-reduced.
(2) $R$ is $\alpha$-symmetric.
(3) $R$ is $\alpha$-semicommutative.
(4) $R$ is $\alpha$-Armendariz.
(5) $R$ is $\alpha$-Armendariz of power series type.
(6) $R$ is $\alpha$-abelian.

Proof. (1) $\Leftrightarrow$ (6) From Lemma 2.12.
(4) $\Leftrightarrow$ (6) Clear from Theorem 2.9.
(3) $\Leftrightarrow$ (6) From Lemma 2.5.
(5) $\Leftrightarrow$ (6) From Proposition 2.11.
(2) $\Rightarrow$ (3) Let $a, b \in R$ with $ab = 0$. By hypothesis, $abc = 0$ implies $acb = 0$ for all $c \in R$. Hence $aRb = 0$ and so $R$ is $\alpha$-semicommutative.
(3) $\Rightarrow$ (2) Assume that $abc = 0$, for any $a, b, c \in R$. Since $R$ is right p.p-ring, $c \in r(ab) = eR$ for some idempotent $e \in R$. Then $c = ec$ and $abe = 0$, so $acbe = 0$. We have already proved that semicommutativity implied being abelian, then $acbe = aecb$. Now $acb = aecb = acbe = 0$. It completes the proof. \(\square\)

Corollary 2.16 Let $R$ be a Baer ring. Then the following are equivalent:
(1) $R$ is $\alpha$-reduced.
(2) $R$ is $\alpha$-symmetric.
(3) $R$ is $\alpha$-semicommutative.
(4) $R$ is $\alpha$-Armendariz.
(5) $R$ is $\alpha$-Armendariz of power series type.
(6) $R$ is $\alpha$-abelian.

One may suspect that if $R$ is an abelian ring, then $R[x, \alpha]$ is abelian also. But this is not the case.

Example 2.17 Let $F$ be any field, $R = \{ \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & 0 & u \end{pmatrix} | a, b, u, v \in F \}$ and $\alpha : R \to R$ be defined by
$$\alpha \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & 0 & u \end{pmatrix} = \begin{pmatrix} u & v & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}, \text{ where } \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & 0 & u \end{pmatrix} \in R$$
Since $R$ is commutative, $R$ is abelian. We claim that $R[x, \alpha]$ is not an abelian ring. Let $e_{ij}$ denote the $4 \times 4$ matrix units having alone 1 as its $(i, j)$-entry and all other entries 0. Consider $e = e_{11} + e_{22}$ and $f = e_{33} + e_{44} \in R$ and $e(x) = e + fx \in R[x; \alpha]$. Then $e(x)^2 = e(x)$, $ef = fe = 0$, $e^2 = e$, $f^2 = f$, $\alpha(e) = f$, $\alpha(f) = e$. An easy calculation reveals that $e(x)e_{12} = e_{12} + e_{34}x$, but $e_{12}e(x) = e_{12}$. Hence $R[x, \alpha]$ is not an abelian ring.

**Lemma 2.18** If $R$ is an $\alpha$-abelian ring, then the idempotents of $R[x, \alpha]$ belong to $R$, therefore $R[x, \alpha]$ is an abelian ring.

**Proof.** Let $R$ be $\alpha$-abelian and $e(x) = \sum_{i=0}^t e_i x^i$ be an idempotent in $R[x, \alpha]$. Since $e(x)^2 = e(x)$, we have

\[
\begin{align*}
e_0^2 &= e_0 \quad (1) \\
e_0 e_1 + e_1 \alpha(e_0) &= e_1 \quad (2) \\
e_0 e_2 + e_1 \alpha(e_1) + e_2 \alpha^2(e_0) &= e_2 \quad (3) \\
&\quad \vdots
\end{align*}
\]

Since $R$ is $\alpha$-abelian, $R$ is abelian, and so every idempotent is central. By Lemma 2.3, $\alpha(e) = e$ for every idempotent $e \in R$. Then (2) becomes $e_0 e_1 + e_1 e_0 = e_1$ and so $e_1 = 0$. Since $e_0$ is central idempotent, (3) becomes $e_0 e_2 + e_2 e_0 = e_2$ and so $e_2 = 0$. Similarly, it can be shown that $e_i = 0$ for $i = 1, 2, \ldots, t$. This completes the proof.

**Lemma 2.19** If $R[x, \alpha]$ is an abelian ring, then $\alpha(e) = e$ for every idempotent $e \in R$.

**Proof.** Since $R[x, \alpha]$ is abelian, we have $f(x)e(x) = e(x)f(x)$ for any $f(x), e(x) \in R[x, \alpha]$. In particular, $xe = ex$ for every idempotent $e \in R$. Hence $xe = ex = \alpha(e)x$ and so $\alpha(e) = e$.

**Lemma 2.20** If $R[x, \alpha]$ is an abelian ring, then the idempotents of $R[x, \alpha]$ belong to $R$.

**Proof.** Similar to the proof of Lemma 2.18.

**Theorem 2.21** If $R$ is an $\alpha$-abelian ring, then $R[x, \alpha]$ is abelian. The converse holds if $R[x, \alpha]$ is a right p.p.-ring.

**Proof.** If $R$ is $\alpha$-abelian, by Lemma 2.18, $R[x, \alpha]$ is abelian. Suppose that $R[x, \alpha]$ be an abelian and right p.p.-ring. It is clear that $ae = ea$ for any $a, e \in R$. Suppose $ab = 0$ for any $a, b \in R$. Since $R$ is right p.p.-ring, we have $b \in r(a) = eR$, $b = eb$. So $\alpha(a)(b) = \alpha(a)(eb) = ace(b) = 0$. Conversely, let $\alpha(a)(b) = 0$. Then $axb = 0$. Since $R[x, \alpha]$ is right p.p.-ring, we have $b \in r_{R[x, \alpha]}(ax) = eR[x, \alpha]$ for some idempotent $e \in R[x, \alpha]$. So $b = eb$, $axe = 0$. By Lemma 2.20, $e \in R$. Hence $ae = 0$ and $ab = ae b = 0$. Therefore $R$ is $\alpha$-abelian.
Lemma 2.22 Let $R$ be an $\alpha$-abelian ring. If for any countable subset $X$ of $R$, $r(X) = eR$, where $e^2 = e \in R$, then
(1) $R[[x, \alpha]]$ is a right p.p.-ring.
(2) If $\alpha$ is an automorphism of $R$, then $R[[x, x^{-1}, \alpha]]$ is a right p.p.-ring.

Proof. Let $a \in R$. Since $\{a\}$ is countable subset of $R$, $r(a) = eR$, i.e., $R$ is a right p.p.-ring. Then from Theorem 2.15, $R$ is Armendariz of power series type. By [11, Theorem 2.11.1(c), Theorem 2.11.2(c)], $R[[x, \alpha]]$ and $R[[x, x^{-1}, \alpha]]$ are right p.p.-rings. 

\[ \square \]

Theorem 2.23 Let $R$ be an $\alpha$-abelian ring. Then we have:
(1) $R$ is a right p.p.-ring if and only if $R[x, \alpha]$ is a right p.p.-ring.
(2) $R$ is a Baer ring if and only if $R[x, \alpha]$ is a Baer ring.
(3) $R$ is a right p.q.-Baer ring if and only if $R[x, \alpha]$ is a right p.q.-Baer ring.
(4) $R$ is a Baer ring if and only if $R[[x, \alpha]]$ is a Baer ring.
(5) $R$ is a Baer ring if and only if $R[[x, x^{-1}, \alpha]]$ is a Baer ring.
(6) $R$ is a right p.p.-ring if and only if $R[[x, x^{-1}, \alpha]]$ is a right p.p.-ring.
(7) $R$ is a Baer ring if and only if $R[[x, x^{-1}, \alpha]]$ is a Baer ring.

Proof. (1) $\Rightarrow$ Let $f(x) = a_0 + a_1x + ... + ax^t \in R[x, \alpha]$. We claim that $r_{R[x, \alpha]}(f(x)) = eR[x, \alpha]$, where $e = e_0e_1...e_t$, $e_i^2 = e_i$ and $r_{R(a_i)} = e_R$, $i = 0, 1, ..., t$. By hypothesis and Lemma 2.3, $f(x)e = a_0e_0e_1...e_t + a_1e_1e_0e_2...e_tx + ... + a_r(e_0e_1...e_{r-1}x^r) = 0$. Then $e \mathcal{R}(x) \subseteq r_{R[x, \alpha]}(f(x))$. Let $g(x) = b_0 + b_1x + ... + b_nx^n \in r_{R[x, \alpha]}(f(x))$. Then $f(x)g(x) = 0$. Since $R$ is an abelian and right p.p.-ring, by Theorem 2.9, $R$ is Armendariz. So $a_ib_j = 0$ and this implies $b_i \in r_{R(a_i)} = e_iR$, and then $b_i = e_i b_i$ for any $i$. Therefore $g(x) = eg(x) \in eR[x, \alpha]$. This completes the proof of (1) $\Rightarrow$.

$\Leftarrow$: Let $a \in R$. Then there exists $e(x)^2 = e(x) \in R[x, \alpha]$ such that $r_{R(x, \alpha)}(a) = e(x)R[x, \alpha]$. Then the constant term, $e_0$, say, of $e(x)$ is non-zero, and $e_0$ is an idempotent in $R$. So $a_0R \subset r_{R(a)}$. Now let $b \in r_{R(a)}$. Since $r_{R(a)} \subseteq r_{R[x, \alpha]}(a)$, $ab = 0$ implies that $b = e(x)b$ and $b \in e_0b$. Hence $r_{R(a)} \subseteq e_0R$, that is, $r_{R(a)} = e_0R$. Therefore $R$ is a right p.p.-ring.

(2) $\Rightarrow$: Since $R$ is Baer, $R$ is a right p.p.-ring. By Lemma 2.5, $R$ is Armendariz. Then from [11, Theorem 2.5.1(c)], $R[x, \alpha]$ is Baer.

$\Leftarrow$: Let $R[x, \alpha]$ be a Baer ring and $X$ be a subset of $R$. There exists $e(x)^2 = e(x) = e_0 + e_1x + ... + e_nx^n \in R[x, \alpha]$ such that $r_{R[x, \alpha]}(X) = e(x)R[x, \alpha]$. We claim that $r_{R(X)}(x) = e_0R$. If $a \in r_{R}(X)$, then $a = e(x)a$ and so $a = e_0a$. Hence $r_{R(X)} \subseteq e_0R$. Since $Xe(x) = 0$, we have $Xe_0 = 0$, that is, $e_0R \subseteq r_{R}(X)$. Then $R$ is a Baer ring.

(3) $\Rightarrow$: Let $f(x) = a_0 + a_1x + ... + a_nx^n \in R[x, \alpha]$. We prove $r_{R[x, \alpha]}(f(x)R[x, \alpha]) = e(x)R[x, \alpha]$, where $e(x) = e_0e_1...e_t$, $r_{R(a,R)} = e_R$. Since $R$ is abelian, for any $h(x) \in R[x, \alpha]$ $f(x)h(x)e(x) = 0$. Then $e(x)R[x, \alpha] \subseteq r_{R[x, \alpha]}(f(x)R[x, \alpha])$. Let $g(x) = b_0 + b_1x + ... + b_nx^n \in r_{R[x, \alpha]}(f(x)R[x, \alpha])$. Then $f(x)g(x)(x) = 0$ and so, $f(x)Rg(x) = 0$. From last equality we have $a_0Rb_0 = 0$. Hence $b_0 \in r_{R(a_0R)} = e_0R$. It follows that $b_0 = e_0b_0$. Also for any $r \in R$, the coefficient of $x$ is equal to $a_0b_1 + a_1\alpha(rb_0)$. 

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Hence $a_0 rb_1 + a_1 \alpha(rb_0) = 0$. Multiplying the equation $a_0 rb_1 + a_1 \alpha(rb_0) = 0$ from the right by $e_0$, we have $a_1 \alpha(rb_0 e_0) = 0$, that is, $a_1 \alpha(rb_0) = 0$. Since $R$ is $\alpha$-abelian, $a_1 rb_0 = 0$. This implies $a_1 Rb_0 = 0$. Then $b_0 \in r_R(a_1 R) = e_1 R$ and $b_1 \in r_R(a_0 R) = e_0 R$. So, $b_0 = e_1 b_0$ and $b_1 = e_0 b_1$. Again for any $r \in R$, $a_0 rb_2 + a_1 rb_1 + a_2 rb_0 = 0$. Multiplying this equality from right by $e_0 e_1$ and using previous results, we have $a_2 rb_0 = 0$. Then $b_1 \in r_R(a_2 R) = e_2 R$. So $b_0 = e_2 b_0$. Continuing this process we have $b_i = e_j b_i$ for any $i, j$. This implies $g(x) = e_0 e_1 \ldots e_j g(x)$. So, $R[x, \alpha]$ is a right p.q.-Baer ring. 

"\Leftarrow": Let $a \in R$. Then $r_{R[x, \alpha]}(aR[x, \alpha]) = e(x)R[x, \alpha]$, where $e(x)^2 = e(x) \in R[x, \alpha]$. By Lemma 2.18, $e(x) = e_0 \in R$. Since $aR[x, \alpha]e(x) = 0$, $aR[x, \alpha]e_0 = 0$ and $aRe_0 = 0$. So, $e_0 R \subset r_R(aR)$. Let $r \in r_R(aR) = r_{R[aR[x, \alpha]]} \subset r_{R[x, \alpha]}(aR[x, \alpha]) = e(x)R[x, \alpha]$. Then $e(x)r = r$. This implies $e_0 r = r$ and so $r \in e_0 R$. Therefore $r_{R[aR[x, \alpha]]} = e_0 R$, i.e., $R$ is a right p.q.-Baer ring.

(4) By Corollary 2.16, every abelian and Baer ring is Armendariz of power series type, so the proof follows from [11, Theorem 2.5 (1)(b)].

(5) By Corollary 2.16, $R$ is $\alpha$-Armendariz, then proof follows from [11, Theorem 2.5 (2)(a)].

(6) Since every $\alpha$-abelian and right p.p.-ring is $\alpha$-Armendariz by Theorem 2.9, the proof follows from [11, Theorem 2.11 (2)(a)].

(7) By Corollary 2.16, every abelian and Baer ring is Armendariz of power series type, it follows from [11, Theorem 2.5 (2)(b)].

\[\Box\]

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