ON PRINCIPALLY QUASI-BAER MODULES

BURCU UNGOR
Department of Mathematics
Ankara University, Ankara, Turkey
Email: burcuungor@gmail.com

NAZIM AGAYEV
Department of Computer Engineering
European University of Lefke, Cyprus
Email: agayev@eul.edu.tr

SAIT HALICI OG LU
Department of Mathematics
Ankara University, Ankara, Turkey
Email: halici@ankara.edu.tr

ABDULLAH HARM ANCI
Maths Department
Hacettepe University, Ankara, Turkey
Email: harmanci@hacettepe.edu.tr

Abstract. Let $R$ be an arbitrary ring with identity and $M$ a right $R$-module with $S = \text{End}_R(M)$. In this paper, we introduce a class of modules that is a generalization of principally quasi-Baer rings and Baer modules. The module $SM$ is called principally quasi-Baer if for any $m \in M$, $l_S(Sm) = Se$ for some $e^2 = e \in S$. It is proved that (1) if $SM$ is regular and semicommutative module or (2) if $MR$ is principally semisimple and $SM$ is abelian, then $SM$ is a principally quasi-Baer module. The connection between a principally quasi-Baer module $SM$ and polynomial extension, power series extension, Laurent polynomial extension, Laurent power series extension of $SM$ is investigated.

1. Introduction

Throughout this paper $R$ denotes an associative ring with identity, and modules will be unitary right $R$-modules. For a module $M$, $S = \text{End}_R(M)$ denotes the ring of right $R$-module endomorphisms of $M$. Then $M$ is a left $S$-module, right $R$-module and $(S, R)$-bimodule. In this work, for any rings $S$ and $R$ and any $(S, R)$-bimodule $M$, $r_M(\cdot)$ and $l_M(\cdot)$ denote the right annihilator of a subset of $M$ in $R$.

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and the left annihilator of a subset of $R$ in $M$, respectively. Similarly, $l_S(.)$ and $r_M(.)$ will be the left annihilator of a subset of $M$ in $S$ and the right annihilator of a subset of $S$ in $M$, respectively. A ring $R$ is reduced if it has no nonzero nilpotent elements. Recently the reduced ring concept was extended to modules by Lee and Zhou in [9], that is, a module $M$ is called reduced if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. A ring $R$ is called semicommutative if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. The module $S M$ is called semicommutative if for any $f \in S$ and $m \in M$, $fm = 0$ implies $f Sm = 0$ (see [3] for details). Baer rings [7] are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. A ring $R$ is said to be right quasi-Baer [5] if the right annihilator of each right ideal of $R$ is generated (as a right ideal) by an idempotent. A ring $R$ is called right principally quasi-Baer [4] if the right annihilator of a principal right ideal of $R$ is generated by an idempotent. An $R$-module $S M$ is called Baer [12] if for all $R$-submodules $N$ of $M$, $l_S(N) = Se$ with $e^2 = e \in S$. The module $S M$ is said to be quasi-Baer if for all fully invariant $R$-submodules $N$ of $M$, $l_S(N) = Se$ with $e^2 = e \in S$. A ring $R$ is called abelian if every idempotent element is central, that is, $ae = ea$ for any $e^2 = e$, $a \in R$. Abelian modules are introduced in the context by Roos in [14] and studied by Goodearl and Boyle [6], Rizvi and Roman [13]. A module $S M$ is called abelian if for any $f \in S$, $e^2 = e \in S$, $m \in M$, we have $fem = efm$. Note that $S M$ is an abelian module if and only if $S$ is an abelian ring. In what follows, by $Z$, $Q$, $Z_n$ and $Z/nZ$ we denote integers, rational numbers, the ring of integers modulo $n$ and the $Z$-module of integers modulo $n$, respectively.

2. Principally Quasi-Baer Modules

Some properties of $R$-modules do not characterize the ring $R$, namely there are reduced $R$-modules but $R$ need not be reduced and there are abelian $R$-modules but $R$ is not an abelian ring. Because of that the investigation of some classes of modules in terms of their endomorphism rings are done by the present authors (see [2] for details). In this section we introduce a class of modules that is a generalization of principally quasi-Baer rings and Baer modules. We prove that some results of principally quasi-Baer rings can be extended to this general setting.

**Definition 2.1.** Let $M$ be an $R$-module with $S = \text{End}_R(M)$. The module $S M$ is called principally quasi-Baer if for any $m \in M$, $l_S(Sm) = Se$ for some $e^2 = e \in S$.

It is straightforward that all Baer, quasi-Baer, semisimple modules are principally quasi-Baer. But a submodule of principally quasi-Baer module may not be principally quasi-Baer. If $e$ is an idempotent element in the ring $R$ and $ere = re$ ($ere = er$) for all $r \in R$, then $e$ is called left (right) semicentral. In the following proposition we prove that idempotents in the definition of principally quasi-Baer modules are right semicentral.

**Proposition 2.2.** Let $M$ be an $R$-module with $S = \text{End}_R(M)$. If $S M$ is a principally quasi-Baer module, then there exists a right semicentral idempotent $e \in S$ such that $l_S(Sm) = Se$ for each $m \in M$.

**Proof.** Let $m \in M$ and $S M$ be a principally quasi-Baer module. By hypothesis, there exists $e^2 = e \in S$ with $l_S(Sm) = Se$. Since $Sef Sm \subseteq SeSm = 0$, we have $Sef Sm = 0$ for all $f \in S$. Hence, $Se \subseteq l_S(Sm) = Se$. Thus, $ef = efe$ for all $f \in S$. □
Theorem 2.3. Let $M$ be an $R$-module with $S = \text{End}_R(M)$. The following are equivalent.

(1) $sM$ is principally quasi-Baer.
(2) The left annihilator of every finitely generated $S$-submodule of $sM$ in $S$ is generated (as a left ideal) by an idempotent.

Proof. (1) $\Rightarrow$ (2) Let $N = \sum_{i=1}^{n} Sm_i$ ($n \in \mathbb{N}$) be a finitely generated $S$-submodule of $M$. Then, $l_S(N) = \bigcap_{i=1}^{n} l_S(Sm_i)$. Since $M$ is principally quasi-Baer, there exist $e_i \in S$ such that $l_S(Sm_i) = Se_i$ for $i = 1, 2, \ldots, n$. So $l_S(N) = \bigcap_{i=1}^{n} Se_i$ with each $e_i$ a right semicentral idempotent of $S$ by Proposition 2.2. Now we show that $Se_1 \cap Se_2 = Se_1e_2$. Since $Se_1e_2 = Se_1e_1$, then $Se_1e_2 \subseteq Se_1 \cap Se_2$. In order to see other inclusion, let $f = f_1e_1 = f_2e_2 \in Se_1 \cap Se_2$ for some $f_1, f_2 \in S$. Then, $f_2 = f_1e_1e_2 = f_2e_2 = f \in Se_1e_2$. Thus, $Se_1 \cap Se_2 \subseteq Se_1e_2$. On the other hand $(e_1e_2)^2 = e_1e_2$, because $e_1$ is right semicentral. In a similar way, we have $l_S(N) = \bigcap_{i=1}^{n} Se_i = S(e_1e_2 \ldots e_n)$ with $(e_1e_2 \ldots e_n)^2 = e_1e_2 \ldots e_n$.

(2) $\Rightarrow$ (1) It is obvious from (2) since every cyclic $S$-submodule of $sM$ is finitely generated. \hfill $\square$

Corollary 2.4. Let $M$ be an $R$-module with $S = \text{End}_R(M)$. If $sM$ is a finitely generated module and $S$ is a principal ideal domain (or a Noetherian ring), then the following are equivalent.

(1) $sM$ is Baer.
(2) $sM$ is quasi-Baer.
(3) $sM$ is principally quasi-Baer.

Proposition 2.5. Let $M$ be an $R$-module with $S = \text{End}_R(M)$. If $sM$ is a principally quasi-Baer module and $N$ a direct summand of $M$, then $TN$ is principally quasi-Baer, where $T = \text{End}_R(N)$.

Proof. Let $N$ be a direct summand of $M$. There exists $e^2 = e \in S$ such that $N = eM$. So the endomorphism ring $T$ of $N$ is $eSe$. Let $n \in N$. Since $sM$ is a principally quasi-Baer module, there exists a right semicentral idempotent $f$ in $S$ such that $l_S(Sn) = Sf$. Hence, $efe$ is an idempotent of $eSe$. We claim that $l_{eSe}(Sn) = (eSe)(efe)$. For any $g \in S$, $gefeTn = 0$, and so $(eSe)(efe) \leq l_{eSe}(Tn)$. On the other hand, let $x \in Sf \cap eSe$. Then, $xTn = xeSn = xeSn \leq xSn = 0$. Hence we have $x \in l_{eSe}(Tn)$. This implies that $Sf \cap eSe \leq l_{eSe}(Tn)$. Now let $yef \in l_{eSe}(Tn)$ with $y \in S$. Since $yefTn = yeSen = yeSn = 0$, we have $yef \in Sf$. It follows that $l_{eSe}(Tn) \leq Sf \cap eSe$. Thus, $l_{eSe}(Tn) = Sf \cap eSe$. In order to see $l_{eSe}(Tn) \leq (eSe)(efe)$, let $x \in l_{eSe}(Tn)$. Then, $x = s_1f = es_2e$ for some $s_1, s_2 \in S$. Notice that $x = xf = s_1f = es_2ef$ and $x = xe = s_1fe = es_2ef$. Hence, $x = xe = xef = s_1fe = es_2ef \in (eSe)(efe)$. Thus, $l_{eSe}(Tn) \leq (eSe)(efe)$. This completes the proof. \hfill $\square$

The direct sum of principally quasi-Baer modules is not principally quasi-Baer as the following example shows.

Example 2.6. Consider $M = \mathbb{Z} \oplus \mathbb{Z}_2$ as a $\mathbb{Z}$-module. Since $\mathbb{Z}$ is a domain and $\mathbb{Z}_2$ is simple, $\mathbb{Z}$ and $\mathbb{Z}_2$ are Baer and so principally quasi-Baer $\mathbb{Z}$-modules. It can
be easily determined that $S = \text{End}_R(M)$ is $\begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix}$. For $m = (2, \overline{1}) \in M$, $l_S(Sm) = \begin{bmatrix} 0 & 0 \\ \mathbb{Z}_2 & 0 \end{bmatrix}$ and $l_S(Sm)$ is not a direct summand of $S$. This implies that $S \mathcal{M}$ is not principally quasi-Baer.

**Theorem 2.7.** Let $M = M_1 \oplus M_2$ be an $R$-module with $S = \text{End}_R(M)$. If $S_i, M_1$ and $S_2, M_2$ are principally quasi-Baer, where $S_1 = \text{End}_R(M_1)$, $S_2 = \text{End}_R(M_2)$ and $\text{Hom}(M_i, M_j) = 0$ for $i \neq j$, $i = j = 1, 2$, then $S \mathcal{M}$ is also principally quasi-Baer.

**Proof.** By hypothesis, $\text{Hom}(M_i, M_j) = 0$ for $i \neq j$, $i = j = 1, 2$, we have $S = S_1 \oplus S_2$. Let $m = (m_1, m_2) \in M$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Since $S, M_i$ is principally quasi-Baer, there exists an idempotent $e_i \in S_i$ with $l_S(S_i m_i) = S_i e_i$ for $i = 1, 2$. On the other hand, we have $l_S(Sm) = l_S(S_1 m_1) \oplus l_S(S_2 m_2)$, and so $l_S(Sm)$ is a direct summand of $S$. □

Let $M$ be an $R$-module with $S = \text{End}_R(M)$. Recall that the submodule $N$ of $M$ is called **fully invariant** if $f(N) \leq N$ for all $f \in S$.

**Proposition 2.8.** Let $M$ be an $R$-module with $S = \text{End}_R(M)$. If $S \mathcal{M}$ is a principally quasi-Baer module, then every principal fully invariant submodule of $M$ is not essential in $M$.

**Proof.** Let $mR$ be a fully invariant submodule of $M$. Since $S \mathcal{M}$ is a principally quasi-Baer module, there exists $e^2 = e \in S$ with $l_S(Sm) = Se$. Then we have $Sm \subseteq r_M(l_S(Sm)) = r_M(Se) = (1-e)M$. Hence, $mR$ is not essential in $M$. □

A module $M$ is said to be **principally semisimple** if every principal submodule is a direct summand of $M$.

**Proposition 2.9.** Let $M$ be an $R$-module with $S = \text{End}_R(M)$. If $M_R$ is principally semisimple and $S \mathcal{M}$ is abelian, then $S \mathcal{M}$ is a principally quasi-Baer module.

**Proof.** If $m \in M$, then by hypothesis $M = mR \oplus K$ for some submodule $K$ of $M$. Let $e$ denote the projection of $M$ onto $mR$. It is routine to show that $l_S(Sm) \leq S(1-e)$. Since $m = em$ and $S \mathcal{M}$ is abelian, we have $S(1-e)Sm = S(1-e)Sem = S(1-e)Sm = 0$. Thus, $S(1-e) \leq l_S(Sm)$. This completes the proof. □

A left $T$-module $M$ is called **regular** (in the sense Zelmanowitz [15]) if for any $m \in M$ there exists a left $T$-homomorphism $M \xrightarrow{\phi} T$ such that $m = \phi(m)m$.

**Proposition 2.10.** Let $M$ be an $R$-module with $S = \text{End}_R(M)$. If $S \mathcal{M}$ is regular and semicommutative, then $S \mathcal{M}$ is a principally quasi-Baer module.

**Proof.** If $m \in M$, then by hypothesis there exists a left $S$-homomorphism $M \xrightarrow{\phi} S$ such that $m = \phi(m)m$. Note that $\phi(m)$ is an idempotent of $S$. We prove $l_S(Sm) = S(1-\phi(m))$. Since $(1-\phi(m))m = 0$ and $S \mathcal{M}$ is semicommutative, we have $(1-\phi(m))Sm = 0$. Then, $S(1-\phi(m)) \leq l_S(Sm)$. Now let $f \in l_S(Sm)$. Hence, $fm = 0$ and so $\phi(fm) = f\phi(m) = 0$. Thus, $f = f - f\phi(m) = f(1-\phi(m)) \in S(1-\phi(m))$. Therefore, $l_S(Sm) \leq S(1-\phi(m))$, and this completes the proof. □

**Lemma 2.11.** Let $M$ be an $R$-module with $S = \text{End}_R(M)$. If $S \mathcal{M}$ is a semicommutative module, then $l_S(Sm) = l_S(m)$ for any $m \in M$. 
Proof. We always have \( l_S(Sm) \subseteq l_S(m) \). Conversely, let \( f \in l_S(m) \). Since \( S \) is a semicommutative module, \( fm \) satisfy \( abc \). Let \( \text{Def} \ 2.12 \). The following are also studied by the present authors in \([1]\) and \([11]\). In our case, we have the following.

\textbf{Definition 2.12.} Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). The module \( S \) is called \textit{symmetric} if for any \( m \in M \) and \( f, g \in S \), \( fgm = 0 \) implies \( gfm = 0 \).

\textbf{Example 2.13.} Let \( M \) be a finite generated torsion \( \mathbb{Z} \)-module. Then \( M \) is isomorphic to the \( \mathbb{Z} \)-module \( (\mathbb{Z}/p_1^{n_1}) \oplus (\mathbb{Z}/p_2^{n_2}) \oplus \cdots \oplus (\mathbb{Z}/p_t^{n_t}) \) where \( p_i (i = 1, \ldots, t) \) are distinct prime numbers and \( n_i (i = 1, \ldots, t) \) are positive integers. \( \text{End}_\mathbb{Z}(M) \) is isomorphic to the commutative ring \( (\mathbb{Z}/p_1^{n_1}) \oplus (\mathbb{Z}/p_2^{n_2}) \oplus \cdots \oplus (\mathbb{Z}/p_t^{n_t}) \). So \( S \) is a symmetric module.

\textbf{Lemma 2.14.} Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). If \( S \) is symmetric, then \( S \) is semicommutative. Converse is true if \( S \) is a principally quasi-Baer module.

\textbf{Proof.} Let \( f \in S \) and \( m \in M \) with \( fm = 0 \). Then for all \( g \in S \), \( gfm = 0 \) implies \( fgm = 0 \). So \( fg = 0 \). Conversely, let \( f, g \in S \) and \( m \in M \) with \( fgm = 0 \). By Lemma 2.11, \( f \in l_S(gm) = l_S(Sgm) = Se \) for some \( e^2 = e \in S \). So \( f = fe \) and \( egm = 0 \). Therefore, \( gfm = gfm = efgm = egf = 0 \) because \( e \) is central.

The proof of Proposition 2.15 is straightforward.

\textbf{Proposition 2.15.} Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). Consider the following conditions for \( f \in S \).

1. \( \text{Ker}f \cap \text{Im}f = 0 \).
2. Whenever \( m \in M \), \( fm = 0 \) if and only if \( \text{Im}f \cap Sm = 0 \).

Then (1) \( \Rightarrow \) (2). If \( S \) is a semicommutative module, then (2) \( \Rightarrow \) (1).

A module \( S \) is called \textit{reduced} if condition (2) of Proposition 2.15 holds for each \( f \in S \).

\textbf{Example 2.16.} Let \( p \) be any prime integer and \( M \) the \( \mathbb{Z} \)-module \( (\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Q} \). Then \( S = \text{End}_\mathbb{R}(M) \) is isomorphic to the matrix ring \( \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a \in \mathbb{Z}_p, b \in \mathbb{Q} \right\} \).

It is evident that \( S \) is a reduced module.

\textbf{Proposition 2.17.} Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). Then the following are equivalent.

1. \( S \) is a reduced module.
2. For any \( f \in S \) and \( m \in M \), \( f^2m = 0 \) implies \( fSm = 0 \).

\textbf{Proof.} It follows from \([9, \text{Lemma 1.2}]\).

\textbf{Lemma 2.18.} Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). If \( S \) is a reduced module, then \( S \) is symmetric. The converse holds if \( S \) is a principally quasi-Baer module.
Proof. For any $f, g \in S$ and $m \in M$ suppose that $fgm = 0$. Then, $(fg)^2(m) = 0$ and by hypothesis $fgSm = 0$. So $fgfm = 0$ and $(gf)^2m = 0$. Then, $gfSm = 0$ implies $gfm = 0$. Therefore, $SM$ is symmetric. Conversely, let $f \in S$ and $m \in M$ with $f^2m = 0$. By Lemma 2.14, $SM$ is semicommutative and from Lemma 2.11, $f \in l_S(fm) = l_S(SSm) = Se$ for some $e^2 = e \in S$. So $f = fe$ and $efm = 0$. Since $SM$ is semicommutative, $efSm = 0$. Then, $fgm = fegm = ef gm = 0$ for any $g \in S$. Therefore, $fSm = 0$ and so $SM$ is a reduced module. 

Next example shows that the reverse implication of the first statement in Lemma 2.18 is not true in general, i.e., there exists a symmetric module which is neither reduced nor principally quasi-Baer.

Example 2.19. Consider the ring

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$$

and the right $R$-module

$$M = \left\{ \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}.$$ 

Let $f \in S$ and $f \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c \\ c & d \end{bmatrix}$. Multiplying the latter by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we have $f \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}$. For any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$, $f \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & ac \\ ac & ad + bc \end{bmatrix}$. Similarly, let $g \in S$ and $g \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c' \\ c' & d' \end{bmatrix}$. Then, $g \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & c' \\ 0 & c' \end{bmatrix}$. For any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$, $g \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & ac' \\ ac' & ad' + bc' \end{bmatrix}$. Then it is easy to check that for any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$,

$$fg \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = f \begin{bmatrix} 0 & ac' \\ ac' & ad' + bc' \end{bmatrix} = \begin{bmatrix} 0 & ac'c \\ ac'c & ad'c + ad'c + bc'c \end{bmatrix}$$

and

$$gf \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = g \begin{bmatrix} 0 & ac \\ ac & ad + bc \end{bmatrix} = \begin{bmatrix} 0 & ac' \\ ac' & ad' + ac'd + bc' \end{bmatrix}$$

Hence, $fg = gf$ for all $f, g \in S$. Therefore, $S$ is commutative and so $SM$ is symmetric. Define $f \in S$ by $f \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$ where $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$. Then, $f \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $f^2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = 0$. Hence, $SM$ is not reduced. Let $m = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. By Lemma 2.14, $SM$ is semicommutative and so by Lemma 2.11, $l_S(Sm) = l_S(m) \neq 0$ since the endomorphism $f$ defined preceding belongs to the $l_S(m)$. The module $M$ is indecomposable as a right $R$-module, therefore $S$ does not have any idempotents other than zero and identity. Hence, $l_S(Sm)$ can not be generated by an idempotent as a left ideal of $S$.

We can summarize the relations between reduced modules, symmetric modules and semicommutative modules by using principally quasi-Baer modules.
Theorem 2.20. Let $M$ be an $R$-module with $S = \text{End}_R(M)$. If $S M$ is a principally quasi-Baer module, then the following conditions are equivalent.

1. $S M$ is a reduced module.
2. $S M$ is a symmetric module.
3. $S M$ is a semicommutative module.

Proof. It follows from Lemma 2.18 and Lemma 2.14. \hfill \Box

In the sequel we investigate extensions of principally quasi-Baer modules. We show that there is a strong connection between principally quasi-Baer modules and polynomial extension, power series extension, Laurent polynomial extension, Laurent power series extension of $M$.

Let $R[x], R[[x]], R[x, x^{-1}]$ and $R[[x, x^{-1}]]$ be the polynomial ring, the power series ring, the Laurent polynomial ring and the Laurent power series ring over $R$, respectively and $M$ an $R$-module with $S = \text{End}_R(M)$. Lee and Zhou [9] introduced the following notations. Consider

$$M[x] = \left\{ \sum_{i=0}^{s} m_i x^i : s \geq 0, m_i \in M \right\},$$

$$M[[x]] = \left\{ \sum_{i=0}^{\infty} m_i x^i : m_i \in M \right\},$$

$$M[x, x^{-1}] = \left\{ \sum_{i=-s}^{t} m_i x^i : s \geq 0, t \geq 0, m_i \in M \right\},$$

$$M[[x, x^{-1}]] = \left\{ \sum_{i=-s}^{\infty} m_i x^i : s \geq 0, m_i \in M \right\}.$$

Each of these is an abelian group under an obvious addition operation. For a module $M$, $M[x]$ is a left $S[x]$-module by the scalar product:

$$m(x) = \sum_{j=0}^{s} m_j x^j \in M[x], \quad \alpha(x) = \sum_{i=0}^{t} f_i x^i \in S[x]$$

$$\alpha(x)m(x) = \sum_{k=0}^{s+t} \left( \sum_{i+j=k} f_i m_j \right) x^k.$$  

With a similar scalar product, $M[[x]], M[x, x^{-1}]$ and $M[[x, x^{-1}]]$ become left modules over $S[[x]], S[x, x^{-1}]$ and $S[[x, x^{-1}]]$, respectively. The modules $M[x]$, $M[[x]], M[x, x^{-1}]$ and $M[[x, x^{-1}]]$ are called the polynomial extension, the power series extension, Laurent polynomial extension and the Laurent power series extension of $M$, respectively. The module $M[x]$ is called a principally quasi-Baer if for any $m(x) \in M[x]$, there exists $e^2 = e \in S[x]$ such that $l_{S[x]}(S[x]m(x)) = S[x]e$. Also $M[[x]], M[x, x^{-1}]$ and $M[[x, x^{-1}]]$ may be defined in a similar way.

Theorem 2.21. Let $M$ be an $R$-module with $S = \text{End}_R(M)$. Then

1. $M[x]$ is a principally quasi-Baer module if and only if $S M$ is a principally quasi-Baer module.
2. If $M[[x]]$ is a principally quasi-Baer module, then $S M$ is a principally quasi-Baer module.
If \( M[x, x^{-1}] \) is a principally quasi-Baer module, then \( sM \) is a principally quasi-Baer module.

(4) If \( M[[x, x^{-1}]] \) is a principally quasi-Baer module, then \( sM \) is a principally quasi-Baer module.

Proof. (1) Assume that \( M[x] \) is a principally quasi-Baer module and \( m \in M \). There exists \( e(x)^2 = e(x) \in S[x] \) such that \( l_{S[x]}(S[x]m) = S[x]e(x) \). Thus, \( S[x]e(x) \subseteq l_{S[x]}(Sm) = l_{S}(Sm[x]). \) For \( f(x) = \sum_{i=0}^{n} f_{i}x^{i} \in l_{S}(Sm)[x], f_{i}Sm = 0 \) for all \( i \geq 0 \). For any \( g(x) = \sum_{j=0}^{k} g_{j}x^{j} \in S[x]m, f(x)g(x) = \sum_{i,j} f_{i}g_{j}x^{i+j} = 0. \) So \( f(x) \in l_{S[x]}(S[x]m) \). Thus, \( l_{S}(Sm[x]) = S[x]e(x) \). Write \( e(x) = \sum_{i=0}^{l} e_{i}x^{i} \), where all \( e_{i} \in l_{S}(Sm) \). Then for any \( h \in l_{S}(Sm) \), we have \( h = h_{1}(x)e(x) \) for some \( h_{1}(x) \in S[x] \). So \( he(x) = h_{1}(x)e(x)e(x) = h_{1}(x)e(x) = h \). It follows that \( h = he_{0} \) for all \( h \in l_{S}(Sm) \). Thus, \( l_{S}(Sm) = S_{e_{0}} \) with \( e_{0}^{2} = e_{0} \). It means that \( S \) is principally quasi-Baer. Conversely, assume \( sM \) is a principally quasi-Baer module. Let \( m(x) = m_{0} + m_{1}x + ... + m_{n}x^{n} \in M[x] \). Then, \( l_{S}(Sm) = S_{e_{i}} \) where \( e_{i} \)'s are right semicentral idempotents for all \( i = 0, 1, ..., n \). Let \( e = e_{0}e_{1}...e_{n} \). Then \( e \) is also a right semicentral in \( S \) and \( Se = \bigcap_{i=0}^{n} l_{S}(Sm_{i}) \). Hence, \( S[x]e \subseteq l_{S[x]}(S[x]m(x)) \).

Note that \( l_{S[x]}(S[x]m(x)) = l_{S[x]}(Sm(x)) \). So, \( S[x]e \subseteq l_{S[x]}(Sm(x)) \). Now, let \( h(x) = h_{0} + h_{1}x + ... + h_{k}x^{k} \in l_{S[x]}(Sm(x)) \). Then, \( (h_{0} + h_{1}x + ... + h_{k}x^{k})S(m_{0} + m_{1}x + ... + m_{n}x^{n}) = 0 \). Hence for any \( \alpha \in S \), we have

\[
\begin{align*}
    h_{0}\alpha m_{0} &= 0 \\
    h_{0}\alpha m_{1} + h_{1}\alpha m_{0} &= 0 \\
    h_{0}\alpha m_{2} + h_{1}\alpha m_{1} + h_{2}\alpha m_{0} &= 0 \\
    &\vdots
\end{align*}
\]

By the first equation, \( h_{0} \in l_{S}(Sm_{0}) = S_{e_{0}} \). It means that \( h_{0} = h_{0}e_{0} \) and \( S_{e_{0}}Sm_{0} = 0 \). For \( f \in S \) consider \( e_{0}f \) instead of \( \alpha \) in (2). Then, \( h_{0}e_{0}f m_{1} + h_{1}e_{0}f m_{0} = h_{0}e_{0}f m_{1} = h_{0}f m_{1} = 0 \). So \( h_{0} \in l_{S}(Sm_{1}) = S_{e_{1}} \). Thus, \( h_{0} \in S_{e_{0}e_{1}} \). Since \( h_{0}Sm_{1} = 0, \) (2) yields \( h_{1}Sm_{0} = 0 \). Hence, \( h_{1} \in l_{S}(Sm_{0}) = S_{e_{0}} \). Now we take \( \alpha = e_{0}e_{1}f \in S \) and apply in (3). Then, \( h_{0}e_{0}e_{1}f m_{2} + h_{1}e_{0}e_{1}f m_{1} + h_{2}e_{0}e_{1}f m_{0} = 0 \). But \( h_{1}e_{0}e_{1}f m_{1} = h_{2}e_{0}e_{1}f m_{0} = 0 \). Hence, \( h_{0}e_{0}e_{1}f m_{2} = h_{2}e_{0}e_{1}f m_{0} = 0 \). So \( h_{0} \in l_{S}(\bigcap_{i=0}^{2} l_{S}(Sm_{i})) = S_{e_{0}e_{1}e_{2}} \). By (3), we have \( h_{1}Sm_{1} + h_{2}Sm_{0} = 0 \). Then we have \( h_{1}e_{0}f m_{1} + h_{2}e_{0}f m_{0} = 0 \). But \( h_{2}e_{0}f m_{0} = 0 \), so \( h_{1}e_{0}f m_{1} = h_{1}f m_{1} = 0 \). Thus, \( h_{1} \in l_{S}(\bigcap_{i=0}^{2} l_{S}(Sm_{i})) = S_{e_{0}e_{1}} \) and \( h_{2}Sm_{0} = 0 \). Hence, \( h_{2} \in l_{S}(Sm_{0}) = S_{e_{0}} \). Continuing this procedure, yields \( h_{i} \in Se \) for all \( i = 1, 2, ..., k \). Hence, \( h(x) \in S[x]e \). Consequently \( S[x]e = l_{S[x]}(S[x]m(x)) \).

(2), (3) and (4) are proved similarly.

References