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ON PRINCIPALLY QUASI-BAER MODULES

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Abstract. Let R be an arbitrary ring with identity and M a right R-module with $S = \mathrm{End}_R(M)$. In this paper, we introduce a class of modules that is a generalization of principally quasi-Baer rings and Baer modules. The module ${}_SM$ is called *principally quasi-Baer* if for any $m \in M$, $l_S(Sm) = Se$ for some $e^2 = e \in S$. It is proved that (1) if ${}_SM$ is regular and semicommutative module or (2) if M_R is principally semisimple and ${}_SM$ is abelian, then ${}_SM$ is a principally quasi-Baer module. The connection between a principally quasi-Baer module ${}_SM$ and polynomial extension, power series extension, Laurent polynomial extension, Laurent power series extension of ${}_SM$ is investigated.

1. Introduction

Throughout this paper R denotes an associative ring with identity, and modules will be unitary right R-modules. For a module M, $S = \mathrm{End}_R(M)$ denotes the ring of right R-module endomorphisms of M. Then M is a left S-module, right R-module and (S, R)-bimodule. In this work, for any rings S and R and any (S, R)-bimodule M, $r_R(.)$ and $l_M(.)$ denote the right annihilator of a subset of M in R

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and the left annihilator of a subset of R in M, respectively. Similarly, $l_S(.)$ and $r_M(.)$ will be the left annihilator of a subset of M in S and the right annihilator of a subset of S in M, respectively. A ring R is *reduced* if it has no nonzero nilpotent elements. Recently the reduced ring concept was extended to modules by Lee and Zhou in [9], that is, a module M is called *reduced* if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. A ring R is called *semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. The module ${}_SM$ is called *semicommutative* if for any $f \in S$ and $m \in M$, $fm = 0$ implies $fSm = 0$ (see [3] for details). *Baer rings* [7] are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. A ring R is said to be *right quasi-Baer* [5] if the right annihilator of each right ideal of R is generated (as a right ideal) by an idempotent. A ring R is called *right principally quasi-Baer* [4] if the right annihilator of a principal right ideal of R is generated by an idempotent. An R-module ${}_SM$ is called *Baer* [12] if for all R-submodules N of M, $l_S(N) = Se$ with $e^2 = e \in S$. The module ${}_SM$ is said to be *quasi-Baer* if for all fully invariant R-submodules N of M, $l_S(N) = Se$ with $e^2 = e \in S$. A ring R is called *abelian* if every idempotent element is central, that is, $ae = ea$ for any $e^2 = e$, $a \in R$. Abelian modules are introduced in the context by Roos in [14] and studied by Goodearl and Boyle [6], Rizvi and Roman [13]. A module ${}_SM$ is called *abelian* if for any $f \in S$, $e^2 = e \in S$, $m \in M$, we have $fem = efm$. Note that ${}_SM$ is an abelian module if and only if S is an abelian ring. In what follows, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}/n\mathbb{Z}$ we denote integers, rational numbers, the ring of integers modulo n and the $\mathbb{Z}$-module of integers modulo n, respectively.

2. Principally Quasi-Baer Modules

Some properties of R-modules do not characterize the ring R, namely there are reduced R-modules but R need not be reduced and there are abelian R-modules but R is not an abelian ring. Because of that the investigation of some classes of modules in terms of their endomorphism rings are done by the present authors (see [2] for details). In this section we introduce a class of modules that is a generalization of principally quasi-Baer rings and Baer modules. We prove that some results of principally quasi-Baer rings can be extended to this general setting.

Definition 2.1. Let M be an R-module with $S = \text{End}_R(M)$. The module ${}_SM$ is called *principally quasi-Baer* if for any $m \in M$, $l_S(Sm) = Se$ for some $e^2 = e \in S$.

It is straightforward that all Baer, quasi-Baer, semisimple modules are principally quasi-Baer. But a submodule of principally quasi-Baer module may not be principally quasi-Baer. If e is an idempotent element in the ring R and $ere = re$ ($ere = er$) for all $r \in R$, then e is called *left (right) semicentral*. In the following proposition we prove that idempotents in the definition of principally quasi-Baer modules are right semicentral.

Proposition 2.2. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is a principally quasi-Baer module, then there exists a right semicentral idempotent $e \in S$ such that $l_S(Sm) = Se$ for each $m \in M$.*

Proof. Let $m \in M$ and ${}_SM$ be a principally quasi-Baer module. By hypothesis, there exists $e^2 = e \in S$ with $l_S(Sm) = Se$. Since $SefSm \subseteq SeSm = 0$, we have $SefSm = 0$ for all $f \in S$. Hence, $Sef \subseteq l_S(Sm) = Se$. Thus, $ef = efe$ for all $f \in S$. $\square$

Theorem 2.3. *Let M be an R-module with $S = End_R(M)$. The following are equivalent.*

(1) *${}_SM$ is principally quasi-Baer.*
(2) *The left annihilator of every finitely generated S-submodule of ${}_SM$ in S is generated (as a left ideal) by an idempotent.*

Proof. (1) ⇒ (2) Let $N = \sum_{i=1}^{n} Sm_i$ $(n \in \mathbb{N})$ be a finitely generated S-submodule of M. Then, $l_S(N) = \bigcap_{i=1}^{n} l_S(Sm_i)$. Since M is principally quasi-Baer, there exist $e_i^2 = e_i \in S$ such that $l_S(Sm_i) = Se_i$ for $i = 1, 2, \ldots, n$. So $l_S(N) = \bigcap_{i=1}^{n} Se_i$ with each e_i a right semicentral idempotent of S by Proposition 2.2. Now we show that $Se_1 \cap Se_2 = Se_1e_2$. Since $Se_1e_2 = Se_1e_2e_1$, then $Se_1e_2 \subseteq Se_1 \cap Se_2$. In order to see other inclusion, let $f = f_1e_1 = f_2e_2 \in Se_1 \cap Se_2$ for some $f_1, f_2 \in S$. Then, $fe_2 = f_1e_1e_2 = f_2e_2 = f \in Se_1e_2$. Thus, $Se_1 \cap Se_2 \subseteq Se_1e_2$. On the other hand $(e_1e_2)^2 = e_1e_2$, because e_1 is right semicentral. In a similar way, we have $l_S(N) = \bigcap_{i=1}^{n} Se_i = S(e_1e_2 \ldots e_n)$ with $(e_1e_2 \ldots e_n)^2 = e_1e_2 \ldots e_n$.
(2) ⇒ (1) It is obvious from (2) since every cyclic S-submodule of ${}_SM$ is finitely generated. □

Corollary 2.4. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is a finitely generated module and S is a principal ideal domain (or a Noetherian ring), then the following are equivalent.*

(1) *${}_SM$ is Baer.*
(2) *${}_SM$ is quasi-Baer.*
(3) *${}_SM$ is principally quasi-Baer.*

Proposition 2.5. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is a principally quasi-Baer module and N a direct summand of M, then ${}_TN$ is principally quasi-Baer, where $T = End_R(N)$.*

Proof. Let N be a direct summand of M. There exists $e^2 = e \in S$ such that $N = eM$. So the endomorphism ring T of N is eSe. Let $n \in N$. Since ${}_SM$ is a principally quasi-Baer module, there exists a right semicentral idempotent f in S such that $l_S(Sn) = Sf$. Hence, efe is an idempotent of eSe. We claim that $l_{eSe}(Tn) = (eSe)(efe)$. For any $g \in S$, $egefeTn = 0$, and so $(eSe)(efe) \leq l_{eSe}(Tn)$. On the other hand, let $x \in Sf \cap eSe$. Then, $xTn = xeSen = xeSn \leq xSn = 0$. Hence we have $x \in l_{eSe}(Tn)$. This implies that $Sf \cap eSe \leq l_{eSe}(Tn)$. Now let $eye \in l_{eSe}(Tn)$ with $y \in S$. Since $eyeTn = eyeSen = eyeSn = 0$, we have $eye \in Sf$. It follows that $l_{eSe}(Tn) \leq Sf \cap eSe$. Thus, $l_{eSe}(Tn) = Sf \cap eSe$. In order to see $l_{eSe}(Tn) \leq (eSe)(efe)$, let $x \in l_{eSe}(Tn)$. Then, $x = s_1f = es_2e$ for some $s_1, s_2 \in S$. Notice that $x = xf = s_1f = es_2ef$ and $x = xe = s_1fe = es_2e$. Hence, $x = xe = xfe = s_1fe = es_2efe \in (eSe)(efe)$. Thus, $l_{eSe}(Tn) \leq (eSe)(efe)$. This completes the proof. □

The direct sum of principally quasi-Baer modules is not principally quasi-Baer as the following example shows.

Example 2.6. Consider $M = \mathbb{Z} \oplus \mathbb{Z}_2$ as a $\mathbb{Z}$-module. Since $\mathbb{Z}$ is a domain and $\mathbb{Z}_2$ is simple, $\mathbb{Z}$ and $\mathbb{Z}_2$ are Baer and so principally quasi-Baer $\mathbb{Z}$-modules. It can be easily determined that $S = \mathrm{End}_{\mathbb{Z}}(M)$ is $\begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix}$. For $m = (2, \overline{1}) \in M$, $l_S(Sm) = \begin{bmatrix} 0 & 0 \\ \mathbb{Z}_2 & 0 \end{bmatrix}$ and $l_S(Sm)$ is not a direct summand of S. This implies that ${}_SM$ is not principally quasi-Baer.

Theorem 2.7. *Let $M = M_1 \oplus M_2$ be an R-module with $S = End_R(M)$. If ${}_{S_1}M_1$ and ${}_{S_2}M_2$ are principally quasi-Baer, where $S_1 = End_R(M_1)$, $S_2 = End_R(M_2)$ and $Hom(M_i, M_j) = 0$ for $i \neq j$, $i = j = 1, 2$, then ${}_SM$ is also principally quasi-Baer.*

Proof. By hypothesis, $Hom(M_i, M_j) = 0$ for $i \neq j$, $i = j = 1, 2$, we have $S = S_1 \oplus S_2$. Let $m = (m_1, m_2) \in M$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Since ${}_{S_i}M_i$ is principally quasi-Baer, there exists an idempotent $e_i \in S_i$ with $l_{S_i}(S_i m_i) = S_i e_i$ for $i = 1, 2$. On the other hand, we have $l_S(Sm) = l_{S_1}(S_1 m_1) \oplus l_{S_2}(S_2 m_2)$, and so $l_S(Sm)$ is a direct summand of S. □

Let M be an R-module with $S = \mathrm{End}_R(M)$. Recall that the submodule N of M is called *fully invariant* if $f(N) \leq N$ for all $f \in S$.

Proposition 2.8. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is a principally quasi-Baer module, then every principal fully invariant submodule of M is not essential in M.*

Proof. Let mR be a fully invariant submodule of M. Since ${}_SM$ is a principally quasi-Baer module, there exists $e^2 = e \in S$ with $l_S(Sm) = Se$. Then we have $Sm \subseteq r_M(l_S(Sm)) = r_M(Se) = (1-e)M$. Hence, mR is not essential in M. □

A module M is said to be *principally semisimple* if every principal submodule is a direct summand of M.

Proposition 2.9. *Let M be an R-module with $S = End_R(M)$. If M_R is principally semisimple and ${}_SM$ is abelian, then ${}_SM$ is a principally quasi-Baer module.*

Proof. If $m \in M$, then by hypothesis $M = mR \oplus K$ for some submodule K of M. Let e denote the projection of M onto mR. It is routine to show that $l_S(Sm) \leq S(1-e)$. Since $m = em$ and ${}_SM$ is abelian, we have $S(1-e)Sm = S(1-e)Sem = S(1-e)eSm = 0$. Thus, $S(1-e) \leq l_S(Sm)$. This completes the proof. □

A left T-module M is called *regular* (in the sense Zelmanowitz [15]) if for any $m \in M$ there exists a left T-homomorphism $M \xrightarrow{\phi} T$ such that $m = \phi(m)m$.

Proposition 2.10. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is regular and semicommutative, then ${}_SM$ is a principally quasi-Baer module.*

Proof. If $m \in M$, then by hypothesis there exists a left S-homomorphism $M \xrightarrow{\phi} S$ such that $m = \phi(m)m$. Note that $\phi(m)$ is an idempotent of S. We prove $l_S(Sm) = S(1-\phi(m))$. Since $(1-\phi(m))m = 0$ and ${}_SM$ is semicommutative, we have $(1-\phi(m))Sm = 0$. Then, $S(1-\phi(m)) \leq l_S(Sm)$. Now let $f \in l_S(Sm)$. Hence, $fm = 0$ and so $\phi(fm) = f\phi(m) = 0$. Thus, $f = f - f\phi(m) = f(1-\phi(m)) \in S(1-\phi(m))$. Therefore, $l_S(Sm) \leq S(1-\phi(m))$, and this completes the proof. □

Lemma 2.11. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is a semicommutative module, then $l_S(Sm) = l_S(m)$ for any $m \in M$.*

Proof. We always have $l_S(Sm) \subseteq l_S(m)$. Conversely, let $f \in l_S(m)$. Since ${}_SM$ is a semicommutative module, $fm = 0$ implies $f \in l_S(Sm)$. □

According to Lambek, a ring R is called *symmetric* [8] if whenever $a, b, c \in R$ satisfy $abc = 0$ implies $cab = 0$. The module M_R is called *symmetric* ([8] and [10]) if whenever $a, b \in R$, $m \in M$ satisfy $mab = 0$, we have $mba = 0$. Symmetric modules are also studied by the present authors in [1] and [11]. In our case, we have the following.

Definition 2.12. Let M be an R-module with $S = \mathrm{End}_R(M)$. The module ${}_SM$ is called *symmetric* if for any $m \in M$ and f, $g \in S$, $fgm = 0$ implies $gfm = 0$.

Example 2.13. Let M be a finitely generated torsion $\mathbb{Z}$-module. Then M is isomorphic to the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p_1^{n_1}) \oplus (\mathbb{Z}/\mathbb{Z}p_2^{n_2}) \oplus ... \oplus (\mathbb{Z}/\mathbb{Z}p_t^{n_t})$ where p_i $(i = 1, ..., t)$ are distinct prime numbers and n_i $(i = 1, ..., t)$ are positive integers. $\mathrm{End}_{\mathbb{Z}}(M)$ is isomorphic to the commutative ring $(\mathbb{Z}_{p_1^{n_1}}) \oplus (\mathbb{Z}_{p_2^{n_2}}) \oplus ... \oplus (\mathbb{Z}_{p_t^{n_t}})$. So ${}_SM$ is a symmetric module.

Lemma 2.14. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is symmetric, then ${}_SM$ is semicommutative. Converse is true if ${}_SM$ is a principally quasi-Baer module.*

Proof. Let $f \in S$ and $m \in M$ with $fm = 0$. Then for all $g \in S$, $gfm = 0$ implies $fgm = 0$. So $fSm = 0$. Conversely, let $f, g \in S$ and $m \in M$ with $fgm = 0$. By Lemma 2.11, $f \in l_S(gm) = l_S(Sgm) = Se$ for some $e^2 = e \in S$. So $f = fe$ and $egm = 0$. Since ${}_SM$ is semicommutative, $egSm = 0$. Therefore, $gfm = gfem = gefm = egfm = 0$ because e is central. □

The proof of Proposition 2.15 is straightforward.

Proposition 2.15. *Let M be an R-module with $S = End_R(M)$. Consider the following conditions for $f \in S$.*
(1) $SKerf \cap Imf = 0$.
(2) *Whenever $m \in M$, $fm = 0$ if and only if $Imf \cap Sm = 0$.*
Then (1) $\Rightarrow$ (2). *If ${}_SM$ is a semicommutative module, then* (2) $\Rightarrow$ (1).

A module ${}_SM$ is called *reduced* if condition (2) of Proposition 2.15 holds for each $f \in S$.

Example 2.16. Let p be any prime integer and M the $\mathbb{Z}$-module $(\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Q}$. Then $S = \mathrm{End}_R(M)$ is isomorphic to the matrix ring $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a \in \mathbb{Z}_p, b \in \mathbb{Q} \right\}$. It is evident that ${}_SM$ is a reduced module.

Proposition 2.17. *Let M be an R-module with $S = End_R(M)$. Then the following are equivalent.*
(1) *${}_SM$ is a reduced module.*
(2) *For any $f \in S$ and $m \in M$, $f^2m = 0$ implies $fSm = 0$.*

Proof. It follows from [9, Lemma 1.2]. □

Lemma 2.18. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is a reduced module, then ${}_SM$ is symmetric. The converse holds if ${}_SM$ is a principally quasi-Baer module.*

Proof. For any $f, g \in S$ and $m \in M$ suppose that $fgm = 0$. Then, $(fg)^2(m) = 0$ and by hypothesis $fgSm = 0$. So $fgfm = 0$ and $(gf)^2m = 0$. Then, $gfSm = 0$ implies $gfm = 0$. Therefore, ${}_SM$ is symmetric. Conversely, let $f \in S$ and $m \in M$ with $f^2m = 0$. By Lemma 2.14, ${}_SM$ is semicommutative and from Lemma 2.11, $f \in l_S(fm) = l_S(Sfm) = Se$ for some $e^2 = e \in S$. So $f = fe$ and $efm = 0$. Since ${}_SM$ is semicommutative, $efSm = 0$. Then, $fgm = fegm = efgm = 0$ for any $g \in S$. Therefore, $fSm = 0$ and so ${}_SM$ is a reduced module. □

Next example shows that the reverse implication of the first statement in Lemma 2.18 is not true in general, i.e., there exists a symmetric module which is neither reduced nor principally quasi-Baer.

Example 2.19. Consider the ring

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$$

and the right R-module

$$M = \left\{ \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

Let $f \in S$ and $f \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c \\ c & d \end{bmatrix}$. Multiplying the latter by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we have $f \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$. For any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$, $f \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & ac \\ ac & ad + bc \end{bmatrix}$. Similarly, let $g \in S$ and $g \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c' \\ c' & d' \end{bmatrix}$. Then, $g \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c' \end{bmatrix}$. For any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$, $g \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & ac' \\ ac' & ad' + bc' \end{bmatrix}$. Then it is easy to check that for any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$,

$$fg \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = f \begin{bmatrix} 0 & ac' \\ ac' & ad' + bc' \end{bmatrix} = \begin{bmatrix} 0 & ac'c \\ ac'c & ad'c + adc' + bc'c \end{bmatrix}$$

and

$$gf \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = g \begin{bmatrix} 0 & ac \\ ac & ad + bc \end{bmatrix} = \begin{bmatrix} 0 & acc' \\ acc' & acd' + ac'd + bcc' \end{bmatrix}$$

Hence, $fg = gf$ for all $f, g \in S$. Therefore, S is commutative and so ${}_SM$ is symmetric. Define $f \in S$ by $f \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$ where $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$. Then, $f \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $f^2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = 0$. Hence, ${}_SM$ is not reduced. Let $m = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. By Lemma 2.14, ${}_SM$ is semicommutative and so by Lemma 2.11, $l_S(Sm) = l_S(m) \neq 0$ since the endomorphism f defined preceding belongs to the $l_S(m)$. The module M is indecomposable as a right R-module, therefore S does not have any idempotents other than zero and identity. Hence, $l_S(Sm)$ can not be generated by an idempotent as a left ideal of S.

We can summarize the relations between reduced modules, symmetric modules and semicommutative modules by using principally quasi-Baer modules.

Theorem 2.20. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is a principally quasi-Baer module, then the following conditions are equivalent.*
(1) *${}_SM$ is a reduced module.*
(2) *${}_SM$ is a symmetric module.*
(3) *${}_SM$ is a semicommutative module.*

Proof. It follows from Lemma 2.18 and Lemma 2.14. □

In the sequel we investigate extensions of principally quasi-Baer modules. We show that there is a strong connection between principally quasi-Baer modules and polynomial extension, power series extension, Laurent polynomial extension, Laurent power series extension of M.

Let $R[x], R[[x]], R[x, x^{-1}]$ and $R[[x, x^{-1}]]$ be the polynomial ring, the power series ring, the Laurent polynomial ring and the Laurent power series ring over R, respectively and M an R-module with $S = \mathrm{End}_R(M)$. Lee and Zhou [9] introduced the following notations. Consider

$$M[x] = \left\{ \sum_{i=0}^{s} m_i x^i \ : \ s \geq 0, m_i \in M \right\},$$
$$M[[x]] = \left\{ \sum_{i=0}^{\infty} m_i x^i \ : m_i \in M \right\},$$
$$M[x, x^{-1}] = \left\{ \sum_{i=-s}^{t} m_i x^i \ : \ s \geq 0, t \geq 0, m_i \in M \right\},$$
$$M[[x, x^{-1}]] = \left\{ \sum_{i=-s}^{\infty} m_i x^i \ : \ s \geq 0, m_i \in M \right\}.$$

Each of these is an abelian group under an obvious addition operation. For a module M, $M[x]$ is a left $S[x]$-module by the scalar product:

$$m(x) = \sum_{j=0}^{s} m_j x^j \ \in M[x] \quad , \quad \alpha(x) = \sum_{i=0}^{t} f_i x^i \ \in S[x]$$

$$\alpha(x) m(x) = \sum_{k=0}^{s+t} \left(\sum_{i+j=k} f_i m_j \right) x^k.$$

With a similar scalar product, $M[[x]]$, $M[x, x^{-1}]$ and $M[[x, x^{-1}]]$ become left modules over $S[[x]]$, $S[x, x^{-1}]$ and $S[[x, x^{-1}]]$, respectively. The modules $M[x]$, $M[[x]]$, $M[x, x^{-1}]$ and $M[[x, x^{-1}]]$ are called the *polynomial extension*, the *power series extension*, *Laurent polynomial extension* and the *Laurent power series extension of* M, respectively. The module $M[x]$ is called a *principally quasi-Baer* if for any $m(x) \in M[x]$, there exists $e^2 = e \in S[x]$ such that $l_{S[x]}(S[x]m(x)) = S[x]e$. Also $M[[x]]$, $M[x, x^{-1}]$ and $M[[x, x^{-1}]]$ may be defined in a similar way.

Theorem 2.21. *Let M be an R-module with $S = End_R(M)$. Then*
(1) *$M[x]$ is a principally quasi-Baer module if and only if ${}_SM$ is a principally quasi-Baer module.*
(2) *If $M[[x]]$ is a principally quasi-Baer module, then ${}_SM$ is a principally quasi-Baer module.*

(3) If $M[x, x^{-1}]$ is a principally quasi-Baer module, then ${}_SM$ is a principally quasi-Baer module.
(4) If $M[[x, x^{-1}]]$ is a principally quasi-Baer module, then ${}_SM$ is a principally quasi-Baer module.

Proof. (1) Assume that $M[x]$ is a principally quasi-Baer module and $m \in M$. There exists $e(x)^2 = e(x) \in S[x]$ such that $l_{S[x]}(S[x]m) = S[x]e(x)$. Thus, $S[x]e(x) \subseteq l_{S[x]}(Sm) = l_S(Sm)[x]$. For $f(x) = \sum_{i=0}^{n} f_i x^i \in l_S(Sm)[x]$, $f_iSm = 0$ for all $i \geq 0$. For any $g(x) = \sum_{j=0}^{k} g_j x^j \in S[x]m, f(x)g(x) = \sum_i \sum_j f_i g_j x^{i+j} = 0$. So $f(x) \in l_{S[x]}(S[x]m)$. Thus, $l_S(Sm)[x] = S[x]e(x)$. Write $e(x) = \sum_{i=0}^{t} e_i x^i$, where all $e_i \in l_S(Sm)$. Then for any $h \in l_S(Sm), h = h_1(x)e(x)$ for some $h_1(x) \in S[x]$. So $he(x) = h_1(x)e(x)e(x) = h_1(x)e(x) = h$. It follows that $h = he_0$ for all $h \in l_S(Sm)$. Thus, $l_S(Sm) = Se_0$ with $e_0^2 = e_0$. It means that ${}_SM$ is principally quasi-Baer. Conversely, assume ${}_SM$ is a principally quasi-Baer module. Let $m(x) = m_0 + m_1x + ... + m_nx^n \in M[x]$. Then, $l_S(Sm_i) = Se_i$ where e_i's are right semicentral idempotents for all $i = 0, 1, ..., n$. Let $e = e_0e_1...e_n$. Then e is also a right semicentral in S and $Se = \bigcap_{i=0}^{n} l_S(Sm_i)$. Hence, $S[x]e \subseteq l_{S[x]}(S[x]m(x))$. Note that $l_{S[x]}(S[x]m(x)) = l_{S[x]}(Sm(x))$. So, $S[x]e \subseteq l_{S[x]}(Sm(x))$. Now, let $h(x) = h_0 + h_1x + ... + h_kx^k \in l_{S[x]}(Sm(x))$. Then, $(h_0 + h_1x + ... + h_kx^k)S(m_0 + m_1x + ... + m_nx^n) = 0$. Hence for any $\alpha \in S$, we have

$$h_0\alpha m_0 = 0 \qquad (1)$$
$$h_0\alpha m_1 + h_1\alpha m_0 = 0 \qquad (2)$$
$$h_0\alpha m_2 + h_1\alpha m_1 + h_2\alpha m_0 = 0 \qquad (3)$$
$$... \qquad ...$$

By the first equation, $h_0 \in l_S(Sm_0) = Se_0$. It means that $h_0 = h_0e_0$ and $Se_0Sm_0 = 0$. For $f \in S$ consider e_0f instead of α in (2). Then, $h_0e_0fm_1 + h_1e_0fm_0 = h_0e_0fm_1 = h_0fm_1 = 0$. So $h_0 \in l_S(Sm_1) = Se_1$. Thus, $h_0 \in Se_0e_1$. Since $h_0Sm_1 = 0$, (2) yields $h_1Sm_0 = 0$. Hence, $h_1 \in l_S(Sm_0) = Se_0$. Now we take $\alpha = e_0e_1f \in S$ and apply in (3). Then, $h_0e_0e_1fm_2 + h_1e_0e_1fm_1 + h_2e_0e_1fm_0 = 0$. But $h_1e_0e_1fm_1 = h_2e_0e_1fm_0 = 0$. Hence, $h_0e_0e_1fm_2 = h_0fm_2 = 0$. So $h_0 \in l_S(\bigcap_{i=0}^{2} l_S(Sm_i)) = Se_0e_1e_2$. By (3), we have $h_1Sm_1 + h_2Sm_0 = 0$. Then we have $h_1e_0fm_1 + h_2e_0fm_0 = 0$. But $h_2e_0fm_0 = 0$, so $h_1e_0fm_1 = h_1fm_1 = 0$. Thus, $h_1 \in l_S(\bigcap_{i=0}^{1} l_S(Sm_i)) = Se_0e_1$ and $h_2Sm_0 = 0$. Hence, $h_2 \in l_S(Sm_0) = Se_0$. Continuing this procedure, yields $h_i \in Se$ for all $i = 1, 2, ..., k$. Hence, $h(x) \in S[x]e$. Consequently $S[x]e = l_{S[x]}(S[x]m(x))$.

(2), (3) and (4) are proved similarly. □

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