On Reduced and Semicommutative Modules

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Abstract

In this paper, various results of reduced and semicommutative rings are extended to reduced and semicommutative modules. In particular, we show: (1) For a principally quasi-Baer module, $M_R$ is semicommutative if and only if $M_R$ is reduced. (2) If $M_R$ is a p.p.-module then $M_R$ is nonsingular.

Key words and phrases: Reduced Rings (Modules), Baer, quasi-Baer and Rings (Modules).

1. Introduction

Throughout this paper all rings $R$ are associative with unity and all modules $M$ are unital right $R$-modules. For a nonempty subset $X$ of a ring $R$, we write $r_R(X) = \{r \in R \mid Xr = 0\}$ and $l_R(X) = \{r \in R \mid rX = 0\}$, which are called the right annihilator of $X$ in $R$ and the left annihilator of $X$ in $R$, respectively. Recall that a ring $R$ is reduced if $R$ has no nonzero nilpotent elements. Observe that reduced rings are abelian (i.e., all idempotents are central).

In [7] Kaplansky introduced Baer rings as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. According to Clark [6], a ring $R$ is said to be quasi-Baer if the right annihilator of each right ideal of $R$ is generated (as a right ideal) by an idempotent. These definitions are left-right symmetric. Recently, Birkenmeier et al. [4] called a ring $R$ a right (resp. left) principally quasi-Baer (or simply, right (resp. left) p.q.-Baer) ring if the right (resp. left) annihilator of a principally right
(resp. left) ideal of $R$ is generated by an idempotent. $R$ is called a $p.q.$-Baer ring if it is both right and left $p.q.$-Baer.

Another generalization of Baer rings is a $p.p.$-ring. A ring $R$ is called a right (resp. left) $p.p.$-ring if the right (resp. left) annihilator of an element of $R$ is generated by an idempotent. $R$ is called a $p.p.$-ring if it is both a right and left $p.p.$-ring.

A ring $R$ is called semicommutative if for every $a \in R$, $r_R(a)$ is an ideal of $R$. (equivalently, for any $a, b \in R$, $ab = 0$ implies $aRb = 0$). Recall from [1] that $R$ is said to satisfy the IFP (insertion of factors property) if $R$ is semicommutative. An idempotent $e \in R$ is called left (resp. right) semicentral if $xe = exe$ (resp. $ex = exe$), for all $x \in R$ ([see, [2]].

According to Lee-Zhou [10], a module $M$ over $R$ is said to be reduced if, for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. It is clear that $R$ is a reduced ring if and only if $R$ is a reduced module.

Lemma [10, Lemma 1.2] The following are equivalent for a module $M_R$:

1. $M_R$ is $\alpha$-reduced.
2. The following three conditions hold: For any $m \in M$ and $a \in R$
   
   (a) $ma = 0$ implies $mRa = mR(a) = 0$.
   (b) $mao(a) = 0$ implies $ma = 0$.
   (c) $ma^2 = 0$ implies $ma = 0$.

In [10] Lee-Zhou introduced Baer, quasi-Baer and the $p.p.$-module as follows:

1. $M_R$ is called Baer if, for any subset $X$ of $M$, $r_R(X) = eR$ where $e^2 = e \in R$.
2. $M_R$ is called quasi-Baer if, for any submodule $N$ of $M$, $r_R(N) = eR$ where $e^2 = e \in R$.
3. $M_R$ is called $p.p.$ if, for any $m \in M$, $r_R(m) = eR$ where $e^2 = e \in R$.

In [8] the module $M_R$ is called principally quasi-Baer ($p.q.$-Baer for short) if, for any $m \in M$, $r_R(mR) = eR$ where $e^2 = e \in R$.

It is clear that $R$ is a right $p.q.$-Baer ring iff $R$ is a $p.q.$-Baer module. If $R$ is a $p.q.$-Baer ring, then for any right ideal $I$ of $R$, $I_R$ is a $p.q.$-Baer module. Every submodule of a $p.q.$-Baer module is a $p.q.$-Baer module. Moreover, every quasi-Baer module is a $p.q.$-Baer, and every Baer module is quasi-Baer. If $R$ is commutative then $M_R$ is $p.p.$-module iff $M_R$ is a $p.q.$-Baer module.
2. Reduced Rings and Modules

We start with the following definition which is defined in [5].

Definition 2.1 A module $M_R$ is called semicommutative if $rR(m)$ is an ideal of $R$ for all $m \in M$. (i.e. for any $m \in M$ and $a \in R$, $ma = 0$ implies $mRa = 0$.)

It is clear that $R$ is semicommutative if and only if $R_M$ is a semicommutative module. Every reduced module is a semicommutative module by [10, Lemma 1.2].

Proposition 2.2 Let $\phi : R \rightarrow S$ be a ring homomorphism and let $M$ be a right $S$-module. Regard $M$ as a right $R$-module via $\phi$. Then we have:

(1) If $M_S$ is a reduced module then $M_R$ is a reduced module.

(2) If $\phi$ is onto, then the converse of the statements in (1) hold.

(3) If $S$ is a reduced ring, then $S_M$ is a reduced as a right $R$-module.

Proof. Straightforward.

Lemma 2.3 If $M_R$ is a semicommutative module, then for any $e^2 = e \in R$, $mea = mae$ for all $m \in M$ and all $a \in R$.

Proof. For $e^2 = e \in R$, $e(1-e) = (1-e)e = 0$. Then for all $m \in M$, $me(1-e) = 0$ and $m(1-e)e = 0$. Since $M_R$ is semicommutative, we have $meR(1-e) = 0$ and $m(1-e)Re = 0$.

Thus for all $a \in R$, $mea(1-e) = 0$ and $m(1-e)ae = 0$. So, $mea = mae$ and $mae = meae$. Hence, $mea = mae$ for all $a \in R$. □

Proposition 2.4 Let $M_R$ be a p.q.-Baer module, then $M_R$ is semicommutative if and only if $M_R$ is reduced.

Proof. Assume $M_R$ is reduced. Then $M_R$ is a semicommutative module by [10, Lemma 1.2].

Conversely, assume $M_R$ is semicommutative. Let $ma = 0$ for $m \in M$ and $a \in R$. Since $M_R$ is p.q.-Baer, $a \in rR(m) = rR(mR) = eR$ where $e^2 = e \in R$. Let $x \in mR \cap Ma$.

Write $x = mr = m'a$ for some $r \in R$ and $m' \in M$. Since $a \in rR(m)$, $a = ea$. Then $x = m'a = m'ea = m'ae$ by Lemma 2.3. So $x = mre = mer = 0$ since $er \in rR(m)$.

Therefore $mR \cap Ma = 0$. Consequently $M_R$ is a reduced module. □
Corollary 2.5 [3, Proposition 1.14.(iv)] If $R$ is a right p.q.-Baer ring, then $R$ satisfies the IFP if and only if $R$ is reduced.

Corollary 2.6 [3, Corollary 1.15] The following are equivalent.

(1) $R$ is a p.q.-Baer ring which satisfies the IFP.

(2) $R$ is a reduced p.q.-Baer ring.

Proposition 2.7 If $M_R$ is a semicommutative module, then

(1) $M_R$ is a Baer module if and only if $M_R$ is a quasi-Baer module.

(2) $M_R$ is a p.p.-module if and only if $M_R$ is a p.q.-Baer module.

Proof. (1) "$\Rightarrow"$ It is clear.

"$\Leftarrow$": Assume $M_R$ is a quasi-Baer module. Let $X$ be any subset of $M_R$. Then $r_R(X) = \bigcap_{x \in X} r_R(x)$. Since $M_R$ is semicommutative, $\bigcap_{x \in X} r_R(x) = \bigcap_{x \in X} r_R(xR)$.

But $M_R$ is quasi-Baer module then $r_R(X) = \bigcap_{x \in X} r_R(xR) = r_R(\sum_{x \in X} xR) = eR$, where $e^2 = e \in R$. Consequently $r_R(X) = eR$, where $e^2 = e \in R$ and hence $M_R$ is a Baer module.

(2) Since $M_R$ is semicommutative, $r_R(m) = r_R(mR)$ for all $m \in M$. Hence proof is clear.

Corollary 2.8 If $R$ is a semicommutative ring, then

(1) $R$ is a Baer ring if and only if $R$ is a quasi-Baer ring.

(2) $R$ is a p.p.-ring if and only if $R$ is a p.q.-Baer ring.

Proposition 2.9 The following conditions are equivalent:

(1) $M_R$ is a p.q.-Baer module.

(2) The right annihilator of every finitely generated submodule is generated (as a right ideal) by an idempotent.

Proof. "(2)$\Rightarrow$(1)" Clear.

"(1)$\Rightarrow$(2)" Assume that $M_R$ is p.q.-Baer and $N = \sum_{i=1}^{k} n_i R$ is a finitely generated submodule of $M_R$. Then $r_R(N) = \bigcap_{i=1}^{k} e_i R$ where $r_R(n_i R) = e_i R$ and $e_i^2 = e_i$. Let $e = e_1 e_2 \ldots e_k$. Then $e$ is a left semicentral idempotent and $\bigcap_{i=1}^{k} e_i R = eR$ since each $e_i$ is a left semicentral idempotent. Therefore, $r_R(N) = eR$. 

288
Corollary 2.10 [3, Proposition 1.7.] The following conditions are equivalent for a ring $R$:
(1) $R$ is a right p.q.-Baer ring.
(2) The right annihilator of every finitely generated ideal of $R$ is generated (as a right ideal) by an idempotent.

Lemma 2.11 Let $M_R$ be a p.p.-module. Then $M_R$ is a reduced module if and only if $M_R$ is a semicommutative module.

Proof. "⇒": It is clear by [10, Lemma 1.2]
"⇐": It follows from Proposition 2.7 and Proposition 2.4

Corollary 2.12 Let $R$ be a right p.p.-ring. Then $R$ is a reduced ring if and only if $R$ is a semicommutative ring.

Proposition 2.13 Let $R$ be an abelian ring. If $M_R$ is a p.p.-module then $M_R$ is a reduced module.

Proof. Let $ma = 0$ for some $m \in M$ and $a \in R$. Then $a \in r_R(m)$. Since $M_R$ is a p.p.-module, $r_R(m) = eR$ where $e^2 = e \in R$. Thus $a = ea$ and $me = 0$. Let $x \in mR \cap Ma$. Write $x = mr = m'a$ for some $r \in R$ and $m' \in M$. Then $x = m'ea = m'e = mre = mer = 0$ since $er \in r_R(m)$. Consequently $M_R$ is a reduced module.

Corollary 2.14 Let $R$ be an abelian ring. If $R$ is a right p.p.-ring then $R$ is a reduced ring.

Proposition 2.15 Let $R$ be an abelian ring and $M_R$ be a p.p.-module. Then $M_R$ is a p.q.-Baer module.

Proof. Let $m \in M$. Since $M_R$ is a p.p.-module, there exists $e^2 = e \in R$ such that $r_R(m) = eR$. It is clear that $r_R(mR) \subseteq r_R(m)$. Let $x \in r_R(m)$. Then $x = ex$ and $me = 0$. For all $r \in R$, $mr = mre = mer = 0$ since $R$ is abelian. Hence, $x \in r_R(mR)$. Consequently $r_R(mR) = r_R(m) = eR$. Therefore $M_R$ is a p.q.-Baer module.

Corollary 2.16 Abelian right p.p.-rings are right p.q.-Baer.

Let $M$ be a module. A submodule $K$ of $M$ is essential in $M$, in case for every submodule $L \leq M$, $K \cap L = 0$ implies $L = 0$. 289
Let $M$ be a right module over a ring $R$. An element $m \in M$ is said to be a *singular element* of $M$ if the right ideal $r_R(m)$ is essential in $R_R$. The set of all singular elements of $M$ is denoted by $Z(M)$. $Z(M)$ is a submodule, called the *singular submodule* of $M$. We say that $M_R$ is a *singular* (resp. *nonsingular*) module if $Z(M) = M$ (resp. $Z(M) = 0$). In particular, we say that $R$ is a *right nonsingular ring* if $Z(R_R) = 0$.

**Proposition 2.17** Every p.p.-module is nonsingular.  
**Proof.** Let $M_R$ be a p.p.-module and $m \in Z(M)$. Then $r_R(m)$ is essential in $R_R$ and there exists $e^2 = e \in R$ such that $r_R(m) = eR$. So $eR$ is essential in $R_R$. But $eR \cap (1-e)R = 0$ for right ideal $(1-e)R$ of $R$. so $(1-e)R = 0$ and hence $e = 1$. Thus $r_R(m) = R$ and so $m = 0$. Therefore $M_R$ is a nonsingular module. \(\square\)

**Corollary 2.18** [9, (7.50)] A right p.p.-ring is right nonsingular.  
The following Lemma given by Lam [9, (7.8) Lemma].

**Lemma** Let $R$ be reduced ring. Then $R$ is right nonsingular.  
Based on this Lemma, one may suspect that, this result true for module case. But the following example eliminates the possibility.

**Example 2.19** The module $(\mathbb{Z}_2)_\mathbb{Z}$ is reduced but not right nonsingular.

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