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On Reduced and Semicommutative Modules

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Abstract

In this paper, various results of reduced and semicommutative rings are extended to reduced and semicommutative modules. In particular, we show: (1) For a principally quasi-Baer module, M_R is semicommutative if and only if M_R is reduced. (2) If M_R is a p.p.-module then M_R is nonsingular.

Key words and phrases: Reduced Rings (Modules), Baer, quasi-Baer and Rings (Modules).

1. Introduction

Throughout this paper all rings R are associative with unity and all modules M are unital right R-modules. For a nonempty subset X of a ring R, we write $r_R(X) = \{r \in R \mid Xr = 0\}$ and $l_R(X) = \{r \in R \mid rX = 0\}$, which are called the right annihilator of X in R and the left annihilator of X in R, respectively. Recall that a ring R is *reduced* if R has no nonzero nilpotent elements. Observe that reduced rings are abelian (i.e., all idempotents are central).

In [7] Kaplansky introduced *Baer* rings as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. Acording to Clark [6], a ring R is said to be *quasi-Baer* if the right annihilator of each right ideal of R is generated (as a right ideal) by an idempotent. These definitions are left-right symmetric. Recently, Birkenmeier et al. [4] called a ring R a *right* (resp. *left*) *principally quasi-Baer* (or simply, *right* (resp. *left*) *p.q.-Baer*) ring if the right (resp. left) annihilator of a principally right

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(resp. left) ideal of R is generated by an idempotent. R is called a *p.q.-Baer* ring if it is both right and left p.q.-Baer.

Another generalization of Baer rings is a p.p.-ring. A ring R is called a *right* (resp. *left*) p.p.-ring if the right (resp. left) annihilator of an element of R is generated by an idempotent. R is called a p.p.-ring if it is both a right and left p.p.-ring.

A ring R is called *semicommutative* if for every $a \in R$, $r_R(a)$ is an ideal of R. (equivalently, for any $a, b \in R$, $ab = 0$ implies $aRb = 0$). Recall from [1] that R is said to satisfy the IFP (*insertion of factors property*) if R is semicommutative. An idempotent $e \in R$ is called *left* (resp. *right*) semicentral if $xe = exe$ (resp. $ex = exe$), for all $x \in R$ ([see, [2]).

According to Lee-Zhou [10], a module M_R is said to be reduced if, for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. It is clear that R is a reduced ring if and only if R_R is a reduced module.

Lemma [10, Lemma 1.2] *The following are equivalent for a module* M_R:

(1) M_R *is* α*-reduced.*

(2) *The following three conditions hold: For any* $m \in M$ *and* $a \in R$

(a) $ma = 0$ *implies* $mRa = mR\alpha(a) = 0$.

(b) $ma\alpha(a) = 0$ *implies* $ma = 0$.

(c) $ma^2 = 0$ *implies* $ma = 0$.

In [10] Lee-Zhou introduced Baer, quasi-Baer and the p.p.-module as follows:

(1) M_R is called *Baer* if, for any subset X of M, $r_R(X) = eR$ where $e^2 = e \in R$.

(2) M_R is called *quasi-Baer* if, for any submodule N of M, $r_R(N) = eR$ where $e^2 = e \in R$.

(3) M_R is called p.p. if, for any $m \in M$, $r_R(m) = eR$ where $e^2 = e \in R$.

In [8] the module M_R is called *principally quasi-Baer* (p.q.-*Baer* for short) if, for any $m \in M$, $r_R(mR) = eR$ where $e^2 = e \in R$.

It is clear that R is a right p.q.-Baer ring iff R_R is a p.q.-Baer module. If R is a p.q.-Baer ring, then for any right ideal I of R, I_R is a p.q.-Baer module. Every submodule of a p.q.-Baer module is p.q.-Baer module. Moreover, every quasi-Baer module is p.q.-Baer, and every Baer module is quasi-Baer. If R is commutative then M_R is p.p.-module iff M_R is p.q.-Baer module.

2. Reduced Rings and Modules

We start with the following definition which is defined in [5].

Deinition 2.1 A module M_R is called *semicommutative* if $r_R(m)$ is an ideal of R for all $m \in M$. (i.e. for any $m \in M$ and $a \in R$, $ma = 0$ implies $mRa = 0$.)

It is clear that R is semicommutative if and only if R_R is a semicommutative module. Every reduced module is a semicommutative module by [10, Lemma 1.2].

Proposition 2.2 *Let $\phi : R \longrightarrow S$ be a ring homomorphism and let M be a right S-module. Regard M as a right R-module via ϕ. Then we have:*

(1) *If M_S is a reduced module then M_R is a reduced module.*

(2) *If ϕ is onto, then the converse of the statements in* (1) *hold.*

(3) *If S is a reduced ring, then S is a reduced as a right R-module.*

Proof. Straightforward.

Lemma 2.3 *If M_R is a semicommutative module, then for any $e^2 = e \in R$, $mea = mae$ for all $m \in M$ and all $a \in R$.*

Proof. For $e^2 = e \in R$, $e(1-e) = (1-e)e = 0$. Then for all $m \in M$, $me(1-e) = 0$ and $m(1-e)e = 0$. Since M_R is semicommutative, we have $meR(1-e) = 0$ and $m(1-e)Re = 0$. Thus for all $a \in R$, $mea(1-e) = 0$ and $m(1-e)ae = 0$. So, $mea = meae$ and $mae = meae$. Hence, $mea = mae$ for all $a \in R$. □

Proposition 2.4 *Let M_R be a p.q.-Baer module, then M_R is semicommutative if and only if M_R is reduced.*

Proof. Assume M_R is reduced. Then M_R is a semicommutative module by [10, Lemma 1.2].

Conversely, assume M_R is semicommutative. Let $ma = 0$ for $m \in M$ and $a \in R$. Since M_R is p.q.-Baer, $a \in r_R(m) = r_R(mR) = eR$ where $e^2 = e \in R$. Let $x \in mR \cap Ma$. Write $x = mr = m'a$ for some $r \in R$ and $m' \in M$. Since $a \in r_R(m)$, $a = ea$. Then $x = m'a = m'ea = m'ae$ by Lemma 2.3. So $x = mre = mer = 0$ since $er \in r_R(m)$. Therefore $mR \cap Ma = 0$. Consequently M_R is a reduced module. □

Corollary 2.5 [3, Proposition 1.14.(iv)] *If R is a right p.q.-Baer ring, then R satisfies the IFP if and only if R is reduced.*

Corollary 2.6 [3, Corollary 1.15] *The following are equivalent.*

(1) *R is a p.q.-Baer ring which satisfies the IFP.*

(2) *R is a reduced p.q.-Baer ring.*

Proposition 2.7 *If M_R is a semicommutative module, then*

(1) *M_R is a Baer module if and only if M_R is a quasi-Baer module.*

(2) *M_R is a p.p.-module if and only if M_R is a p.q.-Baer module.*

Proof. (1) " $\Rightarrow$ " It is clear.

"$\Leftarrow$": Assume M_R is a quasi-Baer module. Let X be any subset of M_R. Then $r_R(X) = \bigcap_{x\in X} r_R(x)$. Since M_R is semicommutative, $\bigcap_{x\in X} r_R(x) = \bigcap_{x\in X} r_R(xR)$. But M_R is quasi-Baer module then $r_R(X) = \bigcap_{x\in X} r_R(xR) = r_R(\sum_{x\in X} xR) = eR$, where $e^2 = e \in R$. Consequently $r_R(X) = eR$, where $e^2 = e \in R$ and hence M_R is a Baer module.

(2) Since M_R is semicommutative, $r_R(m) = r_R(mR)$ for all $m \in M$. Hence proof is clear. □

Corollary 2.8 *If R is a semicommutative ring, then*

(1) *R is a Baer ring if and only if R is a quasi-Baer ring.*

(2) *R is a p.p.-ring if and only if R is a p.q.-Baer ring.*

Proposition 2.9 *The following conditions are equivalent:*

(1) *M_R is a p.q.-Baer module.*

(2) *The right annihilator of every finitely generated submodule is generated (as a right ideal) by an idempotent.*

Proof. "(2)$\Rightarrow$(1)" Clear.

"(1)$\Rightarrow$(2)" Assume that M_R is p.q.-Baer and $N = \sum_{i=1}^{k} n_i R$ is a finitely generated submodule of M_R. Then $r_R(N) = \bigcap_{i=1}^{k} e_i R$ where $r_R(n_i R) = e_i R$ and $e_i^2 = e_i$. Let $e = e_1 e_2 \ldots e_k$. Then e is a left semicentral idempotent and $\bigcap_{i=1}^{k} e_i R = eR$ since each e_i is a left semicentral idempotent. Therefore, $r_R(N) = eR$. □

Corollary 2.10 [3, Proposition 1.7.] *The following conditions are equivalent for a ring R:*

(1) *R is a right p.q.-Baer ring.*

(2) *The right annihilator of every finitely generated ideal of R is generated (as a right ideal) by an idempotent.*

Lemma 2.11 *Let M_R be a p.p.-module. Then M_R is a reduced module if and only if M_R is a semicommutative module.*

Proof. "⇒": It is clear by [10, Lemma 1.2]

"⇐": It follows from Proposition 2.7 and Proposition 2.4 □

Corollary 2.12 *Let R be a right p.p.-ring. Then R is a reduced ring if and only if R is a semicommutative ring.*

Proposition 2.13 *Let R be an abelian ring. If M_R is a p.p.-module then M_R is a reduced module.*

Proof. Let $ma = 0$ for some $m \in M$ and $a \in R$. Then $a \in r_R(m)$. Since M_R is a p.p.-module, $r_R(m) = eR$ where $e^2 = e \in R$. Thus $a = ea$ and $me = 0$. Let $x \in mR \cap Ma$. Write $x = mr = m^{'}a$ for some $r \in R$ and $m^{'} \in M$. Then $x = m^{'}ea = m^{'}ae = mre = mer = 0$ since $er \in r_R(m)$. Consequently M_R is a reduced module. □

Corollary 2.14 *Let R be an abelian ring. If R is a right p.p.-ring then R is a reduced ring.*

Proposition 2.15 *Let R be an abelian ring and M_R be a p.p.-module. Then M_R is a p.q.-Baer module.*

Proof. Let $m \in M$. Since M_R is a p.p.-module, there exists $e^2 = e \in R$ such that $r_R(m) = eR$. It is clear that $r_R(mR) \subseteq r_R(m)$. Let $x \in r_R(m)$. Then $x = ex$ and $me = 0$. For all $r \in R$, $mrx = mrex = merx = 0$ since R is abelian. Hence, $x \in r_R(mR)$. Consequently $r_R(mR) = r_R(m) = eR$. Therefore M_R is a p.q.-Baer module. □

Corollary 2.16 *Abelian right p.p.-rings are right p.q.-Baer.*

Let M be a module. A submodule K of M is *essential* in M, in case for every submodule $L \leq M$, $K \cap L = 0$ implies $L = 0$.

Let M be a right module over a ring R. An element $m \in M$ is said to be a *singular element* of M if the right ideal $r_R(m)$ is essential in R_R. The set of all singular elements of M is denoted by $Z(M)$. $Z(M)$ is a submodule, called the *singular submodule* of M. We say that M_R is a *singular* (resp. *nonsingular*) module if $Z(M) = M$ (resp. $Z(M) = 0$). In particulary, we say that R is a *right nonsingular ring* if $Z(R_R) = 0$.

Proposition 2.17 *Every p.p.-module is nonsingular.*

Proof. Let M_R be a p.p.-module and $m \in Z(M)$. Then $r_R(m)$ is essential in R_R and there exists $e^2 = e \in R$ such that $r_R(m) = eR$. So eR is essential in R_R. But $eR \cap (1-e)R = 0$ for right ideal $(1-e)R$ of R. so $(1-e)R = 0$ and hence $e = 1$. Thus $r_R(m) = R$ and so $m = 0$. Therefore M_R is a nonsingular module. □

Corollary 2.18 [9, (7.50)] *A right p.p.-ring is right nonsingular.*

The following Lemma given by Lam [9, (7.8) Lemma].

Lemma *Let R be reduced ring. Then R is right nonsingular.*

Based on this Lemma, one may suspect that, this result true for module case. But the following example eliminates the possibility.

Example 2.19 The module $(\mathbb{Z}_2)_{\mathbb{Z}}$ is reduced but not right nonsingular.

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