



Ricci solitons on pseudo–Riemannian hypersurfaces of 4–dimensional Minkowski space

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ABSTRACT

In this paper, we get classification theorems for Ricci solitons on pseudo–Riemannian hypersurfaces of a Minkowski space \mathbb{E}_1^4 by taking the potential vector field as the tangent component of the position vector of the pseudo–Riemannian hypersurfaces. First, we obtain the necessary and sufficient condition for a pseudo–Riemannian hypersurface in \mathbb{E}_1^4 which admits such a Ricci soliton. According to the forms of the shape operator of a pseudo–Riemannian hypersurface, we obtain characterization results about Ricci solitons on the pseudo–Riemannian hypersurfaces in \mathbb{E}_1^4 . More precisely, we show that totally umbilical isoparametric hypersurfaces, hyperbolic and a pseudo–spherical cylinder in \mathbb{E}_1^4 admit shrinking Ricci solitons whose potential vector fields are the tangent part of the position vectors in \mathbb{E}_1^4 . Furthermore, we also get that a generalized umbilical Lorentzian hypersurface in \mathbb{E}_1^4 admits a shrinking Ricci soliton. Finally, we obtain that there do not exist Ricci solitons on the Lorentzian hypersurfaces in \mathbb{E}_1^4 whose minimal polynomials have complex roots or real roots with multiplicity three.

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1. Introduction

In the late twentieth century, the concept of Ricci soliton was introduced by Hamilton to prove the Poincaré Conjecture which is related to classify all compact three dimensional manifolds, [18]. The notion of Ricci soliton is related to self similar solutions of the Ricci flow, that is a partial differential equation $\frac{\partial g(t)}{\partial t} = -2\text{Ric}(g(t))$ and it often arises as limit of dilation of singularity in the Ricci flow. For more details, see [6,7]. The topic of Ricci soliton has become more popular and important between mathematicians after G. Perelman used the notion of Ricci soliton to solve the Poincaré conjecture in [21].

We recall the definition of a Ricci soliton on a pseudo–Riemannian manifold as follows:

A smooth vector field ξ on a pseudo–Riemannian manifold (M, g) is said to define a Ricci soliton if it satisfies

$$\frac{1}{2}\mathcal{L}_\xi g + \text{Ric} = \lambda g \quad (1.1)$$

where ξ is a potential vector field, $\mathcal{L}_\xi g$ is the Lie derivative of the metric tensor g with respect to ξ , Ric is the Ricci tensor of (M, g) and λ is a constant. In this context, the Ricci soliton can be considered as a natural generalization of the Einstein metric. Throughout this work, a Ricci soliton on the pseudo–Riemannian manifold (M, g) with a potential vector field ξ is denoted by (M, g, ξ, λ) . A Ricci soliton (M, g, ξ, λ) is called shrinking, steady or expanding if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. A trivial Ricci soliton is one for which ξ is zero or Killing, i.e., $\mathcal{L}_\xi g = 0$ in which case the metric becomes

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Einstein. A Ricci soliton (M, g, ξ, λ) is called a gradient Ricci soliton if its potential vector field ξ is the gradient of some smooth function f on M , that is, $\nabla f = \xi$.

Many mathematicians have obtained important results about the geometry of Ricci soliton from the different point of view such as [3,16,15,5,11,9,12].

The position vector of a pseudo-Riemannian submanifold in the pseudo-Euclidean space is the most elementary and geometric object. There are some research fields in differential geometry related to the position vector of a pseudo-Riemannian submanifold, see [13,14]. In [17], A. Fialkow introduced the notion of concircular vector field v on a Riemannian manifold M as vector field which satisfy

$$\tilde{\nabla}_X v = \mu X, \quad X \in TM \tag{1.2}$$

where $\tilde{\nabla}$ denotes the Levi-Civita connection of M , TM is the tangent bundle of M and μ is a nontrivial function on M . As a particular case for the function $\mu = 1$, a concircular vector field v is called a concurrent vector field. The notion of concircular vector fields on Riemannian manifolds can be extended naturally to concircular vector fields on pseudo-Riemannian manifolds. Since $\tilde{\nabla}_X \mathbf{x} = X$ on a pseudo-Riemannian manifold M , the position vector \mathbf{x} of M is the best known example as concurrent vector field.

A. Barros et al. studied the immersions of Ricci solitons into Riemannian manifolds and they proved that a shrinking Ricci soliton immersed into a space form with constant mean curvature is a Gaussian soliton in [4]. In [8] and [10], B.Y. Chen and S. Deshmukh got the equation for the Ricci tensor of a submanifold in a Riemannian manifold which admits a Ricci soliton whose potential vector field is the tangential part of concurrent vector field and they also completely classified Ricci solitons on Euclidean hypersurfaces whose potential vector fields arisen from the position vector fields of Euclidean hypersurfaces. Then, H. Al-Sodias et al. obtained necessary and sufficient condition for a hypersurface in Euclidean space to be a gradient Ricci soliton in [2]. Moreover, Ş.E. Meriç and E. Kılıç considered under which condition a submanifold of a Ricci soliton is also a Ricci soliton and they gave the relation between intrinsic and extrinsic invariants of a Riemannian submanifold which admits a Ricci soliton in [20].

In this work, we study a Ricci soliton on a pseudo-Riemannian hypersurface M in a 4-dimensional Minkowski space \mathbb{E}_1^4 whose potential vector field is the tangential part of the position vector field of a pseudo-Riemannian hypersurface M in \mathbb{E}_1^4 . First, we find the equation which the Ricci tensor of the pseudo-Riemannian hypersurfaces M in \mathbb{E}_1^4 satisfies to exist such a Ricci soliton on M . From the result in [19], we also know that the shape operator of a pseudo-Riemannian hypersurface M in \mathbb{E}_1^4 can be put into the four different forms, not only diagonalizable one as Riemannian case. In this context, we classify the pseudo-Riemannian hypersurfaces in \mathbb{E}_1^4 with a diagonalizable shape operator and constant mean curvature admitting Ricci solitons $(M, g, \mathbf{x}^T, \lambda)$. Then, we study a Ricci soliton on a Lorentzian hypersurface M in \mathbb{E}_1^4 whose shape operator is not diagonalizable. We obtain that a generalized umbilical Lorentzian hypersurface in \mathbb{E}_1^4 admits a shrinking Ricci soliton $(M, g, \mathbf{x}^T, \lambda)$. Finally, we conclude that there do not exist Ricci solitons $(M, g, \mathbf{x}^T, \lambda)$ on the Lorentzian hypersurfaces M in \mathbb{E}_1^4 whose minimal polynomials of the shape operators have complex roots or real roots with multiplicity three.

2. Preliminaries

Let \mathbb{E}_1^4 be a 4-dimensional real vector space \mathbb{R}^4 with a canonical pseudo-Euclidean metric tensor \tilde{g} of index 1 given by

$$\tilde{g} = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \tag{2.1}$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in \mathbb{R}^4 . Then, $\mathbb{E}_1^4 = (\mathbb{R}^4, \tilde{g})$ is called a 4-dimensional Minkowski space.

The pseudo-Riemannian space forms in \mathbb{E}_1^4 are defined by

$$S_t^n(c^2) = \{\mathbf{x} \in \mathbb{E}_1^4 \mid \tilde{g}(\mathbf{x}, \mathbf{x}) = c^{-2}\} \tag{2.2}$$

$$\mathbb{H}^n(-c^2) = \{\mathbf{x} \in \mathbb{E}_1^4 \mid \tilde{g}(\mathbf{x}, \mathbf{x}) = -c^{-2}\}. \tag{2.3}$$

These spaces are complete and of constant curvature c^2 and $-c^2$, respectively. $S_t^n(c^2)$ and $\mathbb{H}^n(-c^2)$ are called a pseudo-sphere and a hyperbolic space, respectively.

Let $\mathbf{x} : M \rightarrow \mathbb{E}_1^4$ be an isometric immersion from a pseudo-Riemannian hypersurface (M, g) to a pseudo-Euclidean space $(\mathbb{E}_1^4, \tilde{g})$. We denote the Levi-Civita connections of \mathbb{E}_1^4 and M by $\tilde{\nabla}$ and ∇ , respectively. The Gauss and Weingarten formulas of the pseudo-Riemannian hypersurface M in \mathbb{E}_1^4 are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \varepsilon g(AX, Y)N \quad \text{and} \quad \tilde{\nabla}_X N = -AX \tag{2.4}$$

where N is the unit normal vector field with $\varepsilon = \tilde{g}(N, N) = \pm 1$, and A is the shape operator of the pseudo-Riemannian hypersurface M of \mathbb{E}_1^4 in the direction N . The shape operator A is a self-adjoint endomorphism of the tangent space of M , and the shape operator A and the second fundamental form h are related by

$$\tilde{g}(h(X, Y), N) = g(AX, Y). \tag{2.5}$$

The hypersurface M is called totally umbilical if and only if we have

$$h(X, Y) = g(X, Y)H \tag{2.6}$$

for $X, Y \in T_pM$. For $\varepsilon = 1$, M is a Lorentzian hypersurface of \mathbb{E}_1^4 , that is, the induced metric g of M in \mathbb{E}_1^4 is Lorentzian. Otherwise, M is a spacelike hypersurface of \mathbb{E}_1^4 , that is, the induced metric g of M in \mathbb{E}_1^4 is Riemannian.

From the Gauss equation, we get the Ricci tensor of the pseudo-Riemannian hypersurface M in \mathbb{E}_1^4

$$\text{Ric}(X, Y) = 3Hg(AX, Y) - g(AX, AY) \tag{2.7}$$

where $H = \frac{1}{3}\text{tr}A$ is the mean curvature. Also, the Codazzi equation for the pseudo-Riemannian hypersurface M of \mathbb{E}_1^4 is given by

$$(\nabla A)(X, Y) = (\nabla A)(Y, X), \quad X, Y \in T_p(M) \tag{2.8}$$

where $(\nabla A)(X, Y) = \nabla_X(AY) - A(\nabla_X Y)$.

The gradient of a smooth function f on M is a vector field defined by

$$g(\nabla f, X) = df(X) = X(f) \tag{2.9}$$

for any tangent vector field X to M and the Lie derivative of the metric g is defined by

$$(\mathcal{L}_X g)(Y, Z) = X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]) \tag{2.10}$$

for any vector fields X, Y, Z on M . From now on, we will say $\{e_1, e_2, e_3\}$ is an orthonormal frame field defined on M in \mathbb{E}_1^4 if $g(e_1, e_1) = -\varepsilon$, $g(e_2, e_2) = g(e_3, e_3) = 1$ and $g(e_i, e_j) = 0$ for $i \neq j$ and $i, j = 1, 2, 3$. On the other hand, we will say $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal frame field defined on M in \mathbb{E}_1^4 if $g(e_1, e_1) = g(e_2, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0$, $g(e_1, e_2) = -1$ and $g(e_3, e_3) = 1$.

3. A Ricci soliton on a pseudo-Riemannian hypersurface of \mathbb{E}_1^4

In this section, we give some following key results about the Ricci soliton on a pseudo-Riemannian hypersurface of \mathbb{E}_1^4 whose potential vector is the tangential part of the position vector of a pseudo-Riemannian hypersurface in \mathbb{E}_1^4 .

Lemma 3.1. *Let $\mathbf{x} : (M, g) \rightarrow (\mathbb{E}_1^4, \tilde{g})$ be an isometric immersion from a pseudo-Riemannian hypersurface M to the Minkowski space \mathbb{E}_1^4 . Then, the following equations hold*

$$\nabla_X \mathbf{x}^T = X + \varepsilon \rho AX \text{ and } \nabla \rho = -A\mathbf{x}^T \tag{3.1}$$

where \mathbf{x}^T is the tangential part of \mathbf{x} and $\nabla \rho$ is the gradient of the smooth function ρ defined by $\rho = \tilde{g}(\mathbf{x}, N)$.

Proof. Assume that $\mathbf{x} : (M, g) \rightarrow (\mathbb{E}_1^4, \tilde{g})$ is an isometric immersion from a pseudo-Riemannian hypersurface M to a Minkowski space \mathbb{E}_1^4 . Then, we know that the position vector \mathbf{x} can be decomposed as

$$\mathbf{x} = \mathbf{x}^T + \varepsilon \rho N \tag{3.2}$$

where \mathbf{x}^T is the tangential component of \mathbf{x} , $\rho = \tilde{g}(\mathbf{x}, N)$ is a smooth function and $\varepsilon = \tilde{g}(N, N)$.

Since \mathbf{x} is the concurrent vector field on \mathbb{E}_1^4 , that is $\tilde{\nabla}_X \mathbf{x} = X$ for any tangent vector field X to M , from the equation (3.2), we have

$$X = \tilde{\nabla}_X \mathbf{x} = \tilde{\nabla}_X (\mathbf{x}^T + \varepsilon \rho N) = \tilde{\nabla}_X \mathbf{x}^T + \varepsilon (X(\rho)N + \rho \tilde{\nabla}_X N). \tag{3.3}$$

Using the Gauss and Weingarten formulas given by (2.4) in (3.3), we obtain

$$X = \nabla_X \mathbf{x}^T + \varepsilon g(AX, \mathbf{x}^T)N + \varepsilon (X(\rho)N - \rho AX). \tag{3.4}$$

By comparing the tangential and normal part of the equation (3.4), we get

$$\nabla_X \mathbf{x}^T = X + \varepsilon \rho AX \text{ and } X(\rho) = -g(AX, \mathbf{x}^T) \tag{3.5}$$

for any tangent vector field X to M . Thus, we obtain desired result. \square

Now, we get the necessary and sufficient condition to exist a Ricci soliton $(M, g, \mathbf{x}^T, \lambda)$ on the pseudo-Riemannian hypersurface M in \mathbb{E}_1^4 .

Theorem 3.2. A pseudo-Riemannian hypersurface M in the Minkowski space \mathbb{E}_1^4 admits a Ricci soliton $(M, g, \mathbf{x}^T, \lambda)$ if and only if the Ricci tensor of (M, g) satisfies the following equation:

$$\text{Ric}(X, Y) = (\lambda - 1)g(X, Y) - \varepsilon\rho g(AX, Y) \tag{3.6}$$

for any X and Y tangent to M . Moreover, $(M, g, \mathbf{x}^T, \lambda)$ is a gradient Ricci soliton.

Proof. Suppose that $\mathbf{x} : (M, g) \rightarrow (\mathbb{E}_1^4, \tilde{g})$ is an isometric immersion from a pseudo-Riemannian hypersurface M to \mathbb{E}_1^4 and $(M, g, \mathbf{x}^T, \lambda)$ is a Ricci soliton. Then, we recall $\mathbf{x} = \mathbf{x}^T + \varepsilon\rho N$ as explained in the proof of Lemma 3.1. By the virtue of the equations (2.10) and (3.1), we calculate $\mathcal{L}_{\mathbf{x}^T} g$ as

$$(\mathcal{L}_{\mathbf{x}^T} g)(X, Y) = 2g(X, Y) + 2\varepsilon\rho g(AX, Y) \tag{3.7}$$

for any vector fields X, Y on M . Considering this equation with the definition of Ricci soliton (1.1), we get the equation (3.6).

Conversely, it can be shown that if the Ricci tensor of M in \mathbb{E}_1^4 holds the given condition in (3.6), then the equation (1.1) is satisfied for the vector field \mathbf{x}^T and a constant λ . Thus, $(M, g, \lambda, \mathbf{x}^T)$ is a Ricci soliton on a pseudo-Riemannian hypersurface M of \mathbb{E}_1^4 with the potential vector field \mathbf{x}^T .

To prove that $(M, g, \lambda, \mathbf{x}^T)$ is also a gradient Ricci soliton, we choose any smooth function f as $f = \frac{1}{2}\tilde{g}(\mathbf{x}, \mathbf{x})$. Then, we compute

$$X(f) = \tilde{g}(\tilde{\nabla}_X \mathbf{x}, \mathbf{x}) = \tilde{g}(X, \mathbf{x}) = g(X, \mathbf{x}^T) \tag{3.8}$$

which implies $\nabla f = \mathbf{x}^T$. Thus, $(M, g, \lambda, \mathbf{x}^T)$ is a gradient Ricci soliton on the pseudo-Riemannian hypersurface M in \mathbb{E}_1^4 . □

In [19], it was given that a symmetric endomorphism A of a vector space V with a Lorentzian inner product can be put into four different forms. Since the shape operator A of the pseudo-Riemannian hypersurface M in \mathbb{E}_1^4 is a symmetric endomorphism, we consider the four following forms for the shape operator A with respect to orthonormal and pseudo-orthonormal bases of \mathbb{E}_1^4 .

3.1. A Ricci soliton on pseudo-Riemannian hypersurfaces of \mathbb{E}_1^4 with a diagonalizable shape operator

In this section, we are going to consider the case about the Ricci soliton on the pseudo-Riemannian hypersurface in \mathbb{E}_1^4 whose shape operator is diagonalizable.

For using later, we recall the definition of an isoparametric pseudo-Riemannian hypersurface and a generalized constant ratio pseudo-Riemannian hypersurface in \mathbb{E}_1^4 as follows: A pseudo-Riemannian hypersurface is said to be isoparametric if its shape operator is diagonalizable and has constant principal curvatures. A pseudo-Riemannian hypersurface is said to be a generalized constant ratio hypersurface if the tangential part of its position vector is one of principal direction of the shape operator.

Suppose that M is a pseudo-Riemannian hypersurface in \mathbb{E}_1^4 with a diagonalizable shape operator A . Then, we consider an orthonormal tangent frame field $\{e_1, e_2, e_3\}$ on M such that $Ae_i = a_i e_i$, $i = 1, 2, 3$, for smooth functions a_1, a_2 and a_3 . The connection forms corresponding to chosen tangent frame field is defined by $\omega_{ij}(e_k) = \varepsilon_j g(\tilde{\nabla}_{e_k} e_i, e_j)$ for $\varepsilon_j = g(e_j, e_j)$ and $\omega_{ij}(e_k) = -\omega_{ji}(e_k)$. Also, from the Codazzi equation (2.8), we get

$$\begin{aligned} e_i(a_j) &= \varepsilon_j \omega_{ij}(e_j)(a_i - a_j) \\ \omega_{ij}(e_k)(a_i - a_j) &= \omega_{ik}(e_j)(a_i - a_k) \end{aligned} \tag{3.9}$$

where indices i, j, k are distinct and $i, j, k = 1, 2, 3$.

On the other hand, the equation (2.7) gives the components of Ricci tensor of M corresponding to chosen frame field as follows:

$$\begin{aligned} \text{Ric}(e_1, e_1) &= -\varepsilon a_1(a_2 + a_3), & \text{Ric}(e_2, e_2) &= a_2(a_1 + a_3), \\ \text{Ric}(e_3, e_3) &= a_3(a_1 + a_2), & \text{Ric}(e_i, e_j) &= 0, \quad i \neq j \end{aligned} \tag{3.10}$$

for $i, j = 1, 2, 3$. Now, we determine whether there exists a Ricci soliton $(M, g, \mathbf{x}^T, \lambda)$ on the pseudo-Riemannian hypersurface M in \mathbb{E}_1^4 with a diagonalizable shape operator.

Lemma 3.3. Let $\mathbf{x} : (M, g) \rightarrow (\mathbb{E}_1^4, \tilde{g})$ be an isometric immersion from a pseudo-Riemannian hypersurface M with a diagonalizable the shape operator A to the Minkowski space \mathbb{E}_1^4 . Then, $(M, g, \mathbf{x}^T, \lambda)$ is a Ricci soliton if and only if the shape operator A of M satisfies one of the following conditions:

- (i.) $A = c\mathbf{I}$ for a nonzero constant c and 3×3 identity matrix \mathbf{I} .
- (ii.) A has two distinct principal curvatures.

Proof. Suppose that a pseudo-Riemannian hypersurface M of \mathbb{E}_1^4 has a diagonalizable shape operator A and $(M, g, \mathbf{x}^T, \lambda)$ is a Ricci soliton. Thus, we have an orthonormal tangent frame field $\{e_1, e_2, e_3\}$ on M such that $Ae_i = a_i e_i$, $i = 1, 2, 3$, for smooth functions a_1, a_2 and a_3 . For the chosen frame field, the equations (3.9) and (3.10) hold. Since $(M, g, \mathbf{x}^T, \lambda)$ is a Ricci soliton, from the equation (3.6) in Theorem 3.2, we calculate the components of Ricci tensor of M in \mathbb{E}_1^4 as follows:

$$\begin{aligned} \text{Ric}(e_1, e_1) &= \varepsilon(1 - \lambda) + \rho a_1, & \text{Ric}(e_2, e_2) &= \lambda - 1 - \varepsilon \rho a_2, \\ \text{Ric}(e_3, e_3) &= \lambda - 1 - \varepsilon \rho a_3, & \text{Ric}(e_i, e_j) &= 0, \quad i \neq j \end{aligned} \tag{3.11}$$

for $i, j = 1, 2, 3$. Comparing equations (3.10) and (3.11), we have the following system of equation:

$$\lambda - 1 - \varepsilon \rho a_1 = a_1(a_2 + a_3), \tag{3.12}$$

$$\lambda - 1 - \varepsilon \rho a_2 = a_2(a_1 + a_3), \tag{3.13}$$

$$\lambda - 1 - \varepsilon \rho a_3 = a_3(a_1 + a_2). \tag{3.14}$$

Hence, we find the following equations

$$(a_1 - a_2)(a_3 + \varepsilon \rho) = 0 \text{ and } (a_1 - a_3)(a_2 + \varepsilon \rho) = 0 \tag{3.15}$$

which have two different kind of solutions.

If $a_1 = a_2 = a_3$, then the equations in (3.15) are satisfied trivially. Also, from the equation (3.9), we find that all a_i 's are nonzero constants. Remark that when $a_1 = a_2 = a_3 = 0$, the equation (3.10) gives $\text{Ric}(e_i, e_j) = 0$ for $i, j = 1, 2, 3$. We omit this case. Thus, we obtain the result (i). Moreover, $\lambda = 2c^2 + \varepsilon \rho c + 1$.

$a_1 = a_2 = -\varepsilon \rho \neq a_3$ or $a_1 = a_3 = -\varepsilon \rho \neq a_2$ are also the solutions of the given system in (3.15). Thus, we say that the shape operator A of the pseudo-Riemannian hypersurface M in \mathbb{E}_1^4 has two distinct principal curvatures. Moreover, $\lambda = 1 + a_1 a_3$ or $\lambda = 1 + a_2 a_3$.

Conversely, it can be shown that if the shape operator A of the pseudo-Riemannian hypersurface M in \mathbb{E}_1^4 is one of the cases (i) and (ii), then the Ricci tensor of M in \mathbb{E}_1^4 satisfies the equation (3.6) for a suitable constant λ . Thus, $(M, g, \mathbf{x}^T, \lambda)$ is a Ricci soliton on such a pseudo-Riemannian hypersurface M of \mathbb{E}_1^4 . \square

In [1], N. Abe et al. studied the pseudo-Riemannian hypersurfaces in the pseudo-Riemannian real space forms and they got a congruence theorem for isoparametric pseudo-Riemannian hypersurfaces whose shape operators have at most two mutually distinct constant principal curvatures. From the results in [1], we consider the following examples of isoparametric pseudo-Riemannian hypersurfaces in the Minkowski space \mathbb{E}_1^4 :

Example 3.4. A hyperbolic space $\mathbb{H}^3(-c^2)$ with constant curvature $-c^2$ defined by

$$\mathbb{H}^3(-c^2) = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{E}_1^4 \mid -x_1^2 + x_2^2 + x_3^2 + x_4^2 = -c^{-2} \right\}$$

and de Sitter space $\mathbb{S}_1^3(c^2)$ with constant curvature c^2 defined by

$$\mathbb{S}_1^3(c^2) = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{E}_1^4 \mid -x_1^2 + x_2^2 + x_3^2 + x_4^2 = c^{-2} \right\}$$

are totally umbilical isoparametric spacelike and Lorentzian hypersurfaces in \mathbb{E}_1^4 with the shape operator $A = c\mathbf{I}$ for 3×3 identity matrix \mathbf{I} , respectively. It can be easily seen that the Ricci tensor of $\mathbb{H}^3(-c^2)$ and $\mathbb{S}_1^3(c^2)$ in \mathbb{E}_1^4 satisfy (3.6) for $\lambda_1 = c^2 + 1$ or $\lambda_2 = 3c^2 + 1$, respectively. Thus, from Theorem 3.2, $(\mathbb{H}^3(-c^2), g, \mathbf{x}^T, \lambda_1)$ and $(\mathbb{S}_1^3(c^2), g, \mathbf{x}^T, \lambda_2)$ are shrinking Ricci solitons in \mathbb{E}_1^4 .

Example 3.5. Let $\mathbb{H}^2(-c^2) \times \mathbb{E}$ in the Minkowski space \mathbb{E}_1^4 be a spacelike hyperbolic cylinder defined by

$$\mathbb{H}^2(-c^2) \times \mathbb{E} = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{E}_1^4 \mid (x_1, x_2, x_3) \in \mathbb{E}_1^3 \text{ and } -x_1^2 + x_2^2 + x_3^2 = -c^{-2} \right\} \tag{3.16}$$

for a nonzero constant c . The shape operator A has principal curvatures c, c and 0 . It can be easily seen that the Ricci tensor of the spacelike hyperbolic cylinder $\mathbb{H}^2(-c^2) \times \mathbb{E}$ in \mathbb{E}_1^4 satisfies (3.6) with $\lambda = 1$. Hence, from Theorem 3.2, $(\mathbb{H}^2(-c^2) \times \mathbb{E}, g, \mathbf{x}^T, \lambda)$ is a shrinking Ricci soliton in \mathbb{E}_1^4 .

Example 3.6. Let $\mathbb{S}_1^2(c^2) \times \mathbb{E}$ in the Minkowski space \mathbb{E}_1^4 be a pseudo-spherical cylinder defined by

$$\mathbb{S}_1^2(c^2) \times \mathbb{E} = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{E}_1^4 : (x_1, x_2, x_3) \in \mathbb{E}_1^3 \text{ and } -x_1^2 + x_2^2 + x_3^2 = c^{-2} \right\} \tag{3.17}$$

for a nonzero constant c . The shape operator A has principal curvatures c, c and 0 . It can be easily seen that the pseudo-spherical cylinder $\mathbb{S}_1^2(c^2) \times \mathbb{E}$ in \mathbb{E}_1^4 satisfies (3.6) with $\lambda = 1$. Hence, from Theorem 3.2, $(\mathbb{S}_1^2(c^2) \times \mathbb{E}, g, \mathbf{x}^T, \lambda)$ is a shrinking Ricci soliton in \mathbb{E}_1^4 .

By using Lemma 3.3, we get a classification theorem for the Ricci soliton on the pseudo-Riemannian hypersurface in \mathbb{E}_1^4 with a diagonalizable shape operator and constant mean curvature as follows.

Theorem 3.7. *Let M be a pseudo-Riemannian hypersurface of the Minkowski space \mathbb{E}_1^4 with a diagonalizable shape operator and constant mean curvature. Then, $(M, g, \mathbf{x}^T, \lambda)$ is a Ricci soliton if and only if M is an open portion of one of the following the pseudo-Riemannian hypersurfaces of \mathbb{E}_1^4 :*

- (i.) a hyperbolic space $\mathbb{H}^3(-c^2) \subset \mathbb{E}_1^4$ given in Example 3.4,
- (ii.) de Sitter space $\mathbb{S}_1^3(c^2) \subset \mathbb{E}_1^4$ given in Example 3.4,
- (iii.) a hyperbolic cylinder $\mathbb{H}^2(-c^2) \times \mathbb{E}$ given in Example 3.5,
- (iv.) a pseudo-spherical cylinder $\mathbb{S}_1^2(c^2) \times \mathbb{E}$ given in Example 3.6,

Proof. Assume that $(M, g, \mathbf{x}^T, \lambda)$ is a Ricci soliton on the pseudo-Riemannian hypersurface M of \mathbb{E}_1^4 with a diagonalizable shape operator A and a constant mean curvature H . From Lemma 3.3, there are two following cases occur for the shape operator A .

Case a. Suppose that there exists an orthonormal tangent frame field $\{e_1, e_2, e_3\}$ on M such that $Ae_i = ce_i$ for a nonzero constant c . Hence, M is a totally umbilical isoparametric pseudo-Riemannian hypersurface of \mathbb{E}_1^4 . Also, it is a trivial result that M has constant mean curvature $H = c$ in \mathbb{E}_1^4 . Thus, from the result in [1] given as Example 3.4, we get the pseudo-Riemannian hypersurfaces in \mathbb{E}_1^4 given by (i) and (ii) with respect to $\varepsilon = 1$ or $\varepsilon = -1$.

Case b. Suppose that the shape operator A has two distinct principal curvatures. Then, there exists an orthonormal tangent frame field $\{e_1, e_2, e_3\}$ on M such that $Ae_i = a_i e_i$ for some smooth functions a_i . Without loss of generality, we take $a_1 = a_2 \neq a_3$. From the proof of Lemma 3.1, we know that $a_1 = a_2 = -\varepsilon\rho$ and $a_1 a_3 = \lambda - 1$. Also, $H = \frac{1}{3}(2a_1 + a_3)$. In such case, the equations in (3.9) become the followings:

$$e_1(a_1) = e_2(a_1) = e_1(a_3) = e_2(a_3) = 0, \tag{3.18}$$

$$\omega_{13}(e_2) = \omega_{23}(e_1) = \omega_{13}(e_3) = \omega_{23}(e_3) = 0, \tag{3.19}$$

$$e_3(a_1) = -\varepsilon\omega_{13}(e_1)(a_1 - a_3), \tag{3.20}$$

$$a_1 e_3(a_3) = \varepsilon\omega_{13}(e_1)(a_1 - a_3)a_3. \tag{3.21}$$

By using $a_1 = -\varepsilon\rho$ and $e_i(a_1) = 0$ for $i = 1, 2$, we also have $e_i(\rho) = 0$ for $i = 1, 2$. Thus, the equation (3.1) implies

$$a_1 g(e_1, \mathbf{x}^T) = a_1 g(e_2, \mathbf{x}^T) = 0, \tag{3.22}$$

$$e_3(\rho) = -a_3 g(e_3, \mathbf{x}^T). \tag{3.23}$$

For the equation (3.22), if $a_1 = 0$, then from the equation (3.10), it can be easily seen that the Ricci tensor of M in \mathbb{E}_1^4 equals zero. Thus, $a_1 \neq 0$, that is, $g(e_1, \mathbf{x}^T) = g(e_2, \mathbf{x}^T) = 0$. Hence, the position vector \mathbf{x} of M in \mathbb{E}_1^4 is decomposed as

$$\mathbf{x} = g(\mathbf{x}^T, e_3)e_3 + \varepsilon\rho N \tag{3.24}$$

which implies that M is a generalized constant ratio pseudo-Riemannian hypersurface in \mathbb{E}_1^4 . Since M has constant mean curvature in \mathbb{E}_1^4 , we get $e_3(a_3) = -2e_3(a_1)$. Considering the equations (3.20) and (3.21) with this equality, we have

$$\varepsilon\omega_{13}(e_1)(a_1 - a_3)(2a_1 - a_3) = 0. \tag{3.25}$$

When $\omega_{13}(e_1) = 0$, from the equations (3.20), we obtain $a_1 = -\varepsilon\rho$ is a nonzero constant. Thus, the equation (3.23) gives $a_3 = 0$ which means $\lambda = 1$. Remark that for $g(\mathbf{x}^T, e_3) = 0$, we get $\mathbf{x}^T = 0$. In this case, we have a trivial Ricci soliton.

For $a_3 = 2a_1$, $a_1 a_3 = 2a_1^2 = \lambda - 1$ implies that $a_1 = -\varepsilon\rho$ is a nonzero constant. Similarly, we get that $a_3 = 0$ and then $\lambda = 1$. Hence, we have that the shape operator A of M has constant principal curvatures $a_1 = a_2 = -\varepsilon\rho \neq 0$ and $a_3 = 0$. Moreover, $\lambda = 1$. In this case, M is an isoparametric pseudo-Riemannian hypersurface in \mathbb{E}_1^4 . Thus, from the result in [1] given as Example 3.5 and Example 3.6, we get the pseudo-Riemannian hypersurfaces given by (iii) and (iv) with respect to $\varepsilon = 1$ or $\varepsilon = -1$ for $c = \varepsilon\rho$.

The proof of the converse part can be easily seen from Example 3.4–3.6. \square

From Theorem 3.7, we give the following corollary:

Corollary 3.8. *There exists a shrinking Ricci soliton $(M, g, \mathbf{x}^T, \lambda)$ on a pseudo-Riemannian hypersurface M in \mathbb{E}_1^4 with a diagonalizable shape operator and constant mean curvature.*

3.2. A Ricci soliton on Lorentzian hypersurfaces of \mathbb{E}_1^4 with nondiagonalizable shape operator

In this section, we will consider the Ricci soliton on the Lorentzian hypersurface of \mathbb{E}_1^4 having nondiagonalizable shape operator.

From [19], we know that the shape operator A of the Lorentzian hypersurface M in \mathbb{E}_1^4 can be put one of the following form:

- (i.) $A = \begin{pmatrix} a_1 & b_1 & 0 \\ -b_1 & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix}$ with respect to an orthonormal frame field for $b_1 \neq 0$.
- (ii.) $A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 1 \\ -1 & 0 & a_1 \end{pmatrix}$ with respect to a pseudo orthonormal frame field.
- (iii.) $A = \begin{pmatrix} a_1 & 0 & 0 \\ 1 & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix}$ with respect to a pseudo orthonormal frame field.

A Lorentzian hypersurface with a nondiagonalizable shape operator is said to be isoparametric if the minimal polynomial of its shape operator is constant. The minimal polynomial of the shape operator A is a monic polynomial p of least degree such that $p(A) = 0$.

Theorem 3.9. *Let M be a Lorentzian hypersurface in the Minkowski space \mathbb{E}_1^4 with a nondiagonalizable shape operator A . Then, $(M, g, \mathbf{x}^T, \lambda)$ is a Ricci soliton if and only if there exists a pseudo-orthonormal frame field on M to put the shape operator into the following form*

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 1 & a_1 & 0 \\ 0 & 0 & a_1 \end{pmatrix}. \tag{3.26}$$

Moreover, M is an isoparametric Lorentzian hypersurface of \mathbb{E}_1^4 and $(M, g, \mathbf{x}^T, \lambda)$ is a shrinking Ricci soliton.

Proof. Suppose that a Lorentzian hypersurface M of \mathbb{E}_1^4 has a nondiagonalizable shape operator A . Thus, there are three cases for the form of the shape operator A .

Case a. $Ae_1 = a_1e_1 - b_1e_2$, $Ae_2 = b_1e_1 + a_1e_2$ and $Ae_3 = a_2e_3$ with respect to an orthonormal frame field $\{e_1, e_2, e_3\}$ defined on M for some functions a_1, a_2 and $b_1 \neq 0$. Then, the mean curvature of M in \mathbb{E}_1^4 is $H = \frac{1}{3}(2a_1 + a_2)$. From the equation (2.7), we get the components of Ricci tensor of a Lorentzian hypersurface M as follows:

$$\begin{aligned} Ric(e_2, e_2) &= -Ric(e_1, e_1) = a_1^2 + a_1a_2 + b_1^2, \\ Ric(e_1, e_2) &= -a_2b_1, \quad Ric(e_3, e_3) = 2a_1a_2, \\ Ric(e_1, e_3) &= Ric(e_2, e_3) = 0. \end{aligned} \tag{3.27}$$

On the other hand, if M admits a Ricci soliton $(M, g, \mathbf{x}^T, \lambda)$, then we calculate the components of Ricci tensor of M from the equation (3.6) for $\varepsilon = 1$ as follows

$$\begin{aligned} Ric(e_1, e_1) &= -Ric(e_2, e_2) = 1 - \lambda + \rho a_1, \\ Ric(e_3, e_3) &= \lambda - 1 - \rho a_2, \\ Ric(e_1, e_2) &= \rho b_1, \quad Ric(e_1, e_3) = Ric(e_2, e_3) = 0. \end{aligned} \tag{3.28}$$

Considering equations (3.27) and (3.28) together, we obtain the following system of equations:

$$\lambda - 1 - \rho a_1 = a_1^2 + a_1a_2 + b_1^2, \tag{3.29}$$

$$\lambda - 1 - \rho a_2 = 2a_1a_2, \tag{3.30}$$

$$\rho b_1 = -a_2b_1. \tag{3.31}$$

Since $b_1 \neq 0$, the equation (3.31) implies $\rho = -a_2$. Then, from the equation (3.29), we have $\lambda = 1 + a_1^2 + b_1^2$. Using all in the equation (3.30), we get $(a_1 - a_2)^2 + b_1^2 = 0$ that gives $b_1 = 0$. This is a contradiction. Hence, a Lorentzian hypersurface M in \mathbb{E}_1^4 having such a nondiagonalizable shape operator A does not admit the Ricci soliton $(M, g, \mathbf{x}^T, \lambda)$.

Case b. $Ae_1 = a_1e_1 - e_3$, $Ae_2 = a_1e_2$ and $Ae_3 = e_2 + a_1e_3$ with respect to a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ defined on M for a function a_1 . Also, the mean curvature of M in \mathbb{E}_1^4 is $H = a_1$. From the equation (2.7), we compute the coefficients of Ricci tensor of M given by

$$\begin{aligned} Ric(e_1, e_1) &= -1, & Ric(e_1, e_3) &= -a_1, \\ Ric(e_3, e_3) &= -Ric(e_1, e_2) = 2a_1^2, & Ric(e_2, e_2) &= Ric(e_2, e_3) = 0. \end{aligned} \tag{3.32}$$

If $(M, g, \mathbf{x}^T, \lambda)$ is a Ricci soliton on M of \mathbb{E}_1^4 , then using the equation (3.6) for $\varepsilon = 1$, we calculate the components of Ricci tensor of a Lorentzian hypersurface M as follows:

$$\begin{aligned} Ric(e_1, e_1) &= Ric(e_2, e_2) = Ric(e_2, e_3) = 0, & Ric(e_1, e_3) &= \rho \\ Ric(e_1, e_2) &= -Ric(e_3, e_3) = 1 - \lambda + \rho a_1. \end{aligned} \tag{3.33}$$

Since $Ric(e_1, e_1) = -1 \neq 0 = Ric(e_1, e_1)$, there does not exist a Ricci soliton $(M, g, \mathbf{x}^T, \lambda)$ on a Lorentzian hypersurface M of \mathbb{E}_1^4 having such a nondiagonalizable shape operator A .

Case c. $Ae_1 = a_1e_1 + e_2$, $Ae_2 = a_1e_2$ and $Ae_3 = a_2e_3$ with respect to a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ defined on M for smooth functions a_1 and a_2 . Then, the mean curvature of M in \mathbb{E}_1^4 is $H = \frac{1}{3}(2a_1 + a_2)$. By calculation from the equation (2.7), we obtain the components of Ricci tensor of M given by

$$\begin{aligned} Ric(e_1, e_1) &= -a_1 - a_2 - \frac{1}{2}, & Ric(e_1, e_3) &= Ric(e_2, e_3) = 0, & Ric(e_2, e_2) &= a_1 + \frac{1}{2} \\ Ric(e_1, e_2) &= -a_1(a_1 + a_2), & Ric(e_3, e_3) &= 2a_1a_2. \end{aligned} \tag{3.34}$$

Since $(M, g, \mathbf{x}^T, \lambda)$ is a Ricci soliton on a Lorentzian hypersurface of M of \mathbb{E}_1^4 , that is $\varepsilon = 1$, the equation (3.6) gives the following equations:

$$\begin{aligned} Ric(e_1, e_1) &= \rho, & Ric(e_1, e_3) &= Ric(e_2, e_2) = Ric(e_2, e_3) = 0 \\ Ric(e_1, e_2) &= 1 - \lambda + \rho a_1, & Ric(e_3, e_3) &= \lambda - 1 - \rho a_2. \end{aligned} \tag{3.35}$$

Considering equations (3.34) and (3.35), we get $a_1 = -\frac{1}{2}$ and $\rho = -a_2$ and the following equation system is also obtained

$$\lambda - 1 - \rho a_2 = 2a_1a_2, \tag{3.36}$$

$$\lambda - 1 - \rho a_1 = a_1^2 + a_1a_2, \tag{3.37}$$

which give $a_1 = a_2$ and $\lambda = a_1^2 + 1 = \frac{5}{4} > 0$. Thus, $(M, g, \mathbf{x}^T, \lambda)$ is a shrinking Ricci soliton. Hence, M is an isoparametric Lorentzian hypersurface of \mathbb{E}_1^4 . \square

In [19], M. Magid studied Lorentzian isoparametric hypersurfaces in \mathbb{E}_1^4 and they got classification theorem for Lorentzian hypersurfaces according to the principal curvatures and the minimal polynomials of the shape operators. From the result in [19], we consider the following examples of Lorentzian isoparametric hypersurfaces in the Minkowski space \mathbb{E}_1^4 :

Example 3.10. Let $\alpha(s)$ be a null curve in \mathbb{E}_1^4 with a pseudo-orthonormal frame $\{X(s), Y(s), Z(s), W(s)\}$ such that

$$\begin{aligned} \dot{\alpha}(s) &= X(s), \\ \dot{Z}(s) &= -aX(s) - B(s)Y(s) \end{aligned} \tag{3.38}$$

where $B(s) \neq 0$ and a is a nonzero constant. The Lorentzian hypersurface M in \mathbb{E}_1^4 is locally defined as follows:

$$M : \mathbf{x}(s, u, v) = \alpha(s) + uY(s) + vW(s) + \left(\sqrt{\frac{1}{a^2} - v^2} - \frac{1}{a} \right) Z(s) \tag{3.39}$$

which is called as a generalized umbilical hypersurface. The unit normal vector N is

$$N(s, u, v) = -auY(s) - \sqrt{1 - a^2v^2}Z(s) - avW(s) \tag{3.40}$$

and the shape operator A of M in the direction N can be put in the following form with respect to a suitable chosen pseudo-orthonormal frame field

$$A = \begin{pmatrix} a & 0 & 0 \\ k(s) & a & 0 \\ 0 & 0 & a \end{pmatrix}, \quad k(s) \neq 0. \tag{3.41}$$

The minimal polynomial of A is $(x - a)^2$. From Theorem 3.9, the Ricci tensor of the generalized umbilical Lorentzian hypersurface M in \mathbb{E}_1^4 given by (3.39) satisfies (3.6) with $\lambda = a^2 + 1 > 0$. Hence, $(M, g, \mathbf{x}^T, \lambda)$ is a shrinking Ricci soliton in \mathbb{E}_1^4 .

Example 3.11. Let $\alpha(s)$ be a null curve in \mathbb{E}_1^4 with a pseudo-orthonormal frame $\{X(s), Y(s), Z(s), W(s)\}$ such that

$$\begin{aligned} \dot{\alpha}(s) &= X(s), \\ \dot{Z}(s) &= -B(s)Y(s), \quad B(s) \neq 0. \end{aligned} \tag{3.42}$$

The Lorentzian hypersurface M in \mathbb{E}_1^4 is locally defined as follows:

$$M : \mathbf{x}(s, u, v) = \alpha(s) + uY(s) + vW(s) \tag{3.43}$$

which is called as a generalized cylinder of type I. The unit normal vector N is $N = Z(s)$ and the shape operator A of M in the direction N can be put in the following form with respect to a suitable chosen pseudo-orthonormal frame field on M

$$A = \begin{pmatrix} 0 & 0 & 0 \\ k(s) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad k(s) \neq 0. \tag{3.44}$$

Also, we have that the minimal polynomial of A is x^2 . From a direct calculation, it can be easily seen that the Ricci tensor Ric of hypersurface M in \mathbb{E}_1^4 equals zero, that is, $\mathcal{L}_{\mathbf{x}^T} g = g$. Thus, M has a conformal killing vector field \mathbf{x}^T .

From the proof of Theorem 3.9, we obtain that the minimal polynomial of A is $p(x) = (x - a_1)^2$ with a nonzero constant a_1 . Thus, using Theorem 4.5 in [19], we state the following Theorem.

Theorem 3.12. *Let M be a Lorentzian hypersurface in the Minkowski space \mathbb{E}_1^4 with a nondiagonalizable shape operator. Then, $(M, g, \mathbf{x}^T, \lambda)$ is a Ricci soliton if and only if M is an open portion of a generalized umbilical Lorentzian hypersurface of \mathbb{E}_1^4 given in Example 3.10. Moreover, it is a shrinking Ricci soliton.*

From Theorem 3.9, we state the following corollary:

Corollary 3.13. *There do not exist Ricci soliton $(M, g, \mathbf{x}^T, \lambda)$ on a Lorentzian hypersurface M in \mathbb{E}_1^4 whose minimal polynomial of shape operator has complex roots or real roots with multiplicity with three.*

Remark 3.14. In the proof of Theorem 3.9, it can be easily seen that $a_1 = 0$ for $\lambda = 1$. Thus, $\text{Ric}(X, Y) = 0$ for any tangent vectors X, Y of a Lorentzian hypersurface M in \mathbb{E}_1^4 . Moreover, $\mathcal{L}_{\mathbf{x}^T} g = g$, which means \mathbf{x}^T is conformal killing vector field. Hence, M is an open portion of a generalized cylinder of type I in \mathbb{E}_1^4 given in Example 3.11.

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