



Surfaces in Robertson-Walker Space-Times with Positive Relative Nullity

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Abstract

In this article, we study space-like and time-like surfaces in a Robertson-Walker space-time, denoted by $L_1^4(f, c)$, having positive relative nullity. First, we give the necessary and sufficient conditions for such space-like and time-like surfaces in $L_1^4(f, c)$. Then, we obtain the local classification theorems for space-like and time-like surfaces in $L_1^4(f, 0)$ with positive relative nullity, where the second factor is 3-dimensional Euclidean space. Finally, we consider the space-like and time-like surfaces in $\mathbb{E}_1^1 \times \mathbb{S}^3$ and $\mathbb{E}_1^1 \times \mathbb{H}^3$ with positive relative nullity. These are the special spaces of $L_1^4(f, c)$ when the warping function f is a constant function, with $c = 1$ for $\mathbb{E}_1^1 \times \mathbb{S}^3$ and $c = -1$ for $\mathbb{E}_1^1 \times \mathbb{H}^3$.

Keywords Robertson-Walker space-time · Lorentzian product spaces · Positive relative nullity

Mathematics Subject Classification 53C40 · 53C42

1 Introduction

The relative null space of a submanifold of a pseudo-Riemannian manifold at a point is the subspace of its tangent space on which the second fundamental form vanishes identically. Then, a submanifold in a pseudo-Riemannian manifold is said to have

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positive relative nullity if, at every point on the submanifold, the dimension of the relative null space is greater than zero.

There are several works related to submanifolds with the relative null space and positive relative nullity in different ambient spaces. M. Dajczer and D. Gromoll introduced the concept of submanifolds with positive relative nullity and they obtained the necessary and sufficient conditions for a spherical submanifold to have positive relative nullity in [6]. B.-Y. Chen and J. Van der Veken provided a complete classification of marginally trapped surfaces with positive relative nullity in Lorentzian space-forms, namely the Minkowski space \mathbb{E}_1^4 , the de Sitter space \mathbb{S}_1^4 and the anti-de Sitter space \mathbb{H}_1^4 in [3]. Also, M. Dajczer and R. Tojeiro gave some results about submanifolds of space-forms having positive relative nullity, [7]. Additionally, the authors [2] studied marginally trapped surfaces with positive relative nullity in 4-dimensional pseudo-Riemannian space-forms with index 2, namely pseudo-Euclidean space \mathbb{E}_2^4 , the pseudo-sphere \mathbb{S}_2^4 and the pseudo-hyperbolic space \mathbb{H}_2^4 .

Another example of the Lorentzian space is the family of Robertson-Walker space-times which is an important family of cosmological models in general relativity. The family of Robertson Walker space times includes the de Sitter, Minkowski, and the anti-de Sitter space times and also Friedmann's cosmological models. In physics, Robertson-Walker space-times explains homogeneous, isotropic expanding and contracting universes, see [13] and [10]. Motivated by their physical importance and rich geometric structure, Robertson-Walker space-times have also attracted attention in differential geometry. In this context, some studies have focused on classifying surfaces or hypersurfaces of Robertson-Walker space-times having different geometrical properties such as [1, 4, 8]. Especially, within the context of positive relative nullity B.-Y. Chen and J. Van der Veken [4] investigated a marginally trapped surface with positive relative nullity in a Robertson Walker space-time and they proved that a Robertson Walker space-time, which contains no open subsets of constant curvature, does not allow for the existence of a marginally trapped surface with positive relative nullity.

The aim of this paper is to examine space-like and time-like surfaces in a Robertson-Walker space-time $L_1^4(f, c)$ that possess positive relative nullity, without assuming the marginally trapped condition. First, we give the necessary and sufficient conditions for such space-like and time-like surfaces in $L_1^4(f, c)$ and we also get the forms of Levi-Civita connections of them. Then, we focus on $c = 0$, that is, $L_1^4(f, 0) = I_1^1 \times_f \mathbb{E}^3$, and we obtain the classification theorems for space-like and time-like surfaces in $L_1^4(f, 0)$ having positive relative nullity.

Finally, we consider the case where the warping function f is constant. In this case, the space-time becomes to the Lorentzian product spaces $\mathbb{E}_1^1 \times \mathbb{S}^3$ and $\mathbb{E}_1^1 \times \mathbb{H}^3$, corresponding to $c = 1$ and $c = -1$, respectively. We end the paper with the classification results for space-like and time-like surfaces with positive relative nullity in both spaces.

2 Preliminaries

In this section, we give a brief summary of basic facts and equations of the theory of submanifolds and Robertson Walker Spaces, (see [5, 10]).

2.1 Robertson-Walker Spaces

Let $R^{n-1}(c)$ denote the $n - 1$ dimensional Riemannian space-form with the constant sectional curvature c , i.e.,

$$R^{n-1}(c) = \begin{cases} \mathbb{S}^{n-1} & \text{if } c = 1, \\ \mathbb{E}^{n-1} & \text{if } c = 0, \\ \mathbb{H}^{n-1} & \text{if } c = -1 \end{cases}$$

and g_c stand for its metric tensor and $\nabla^{R^{n-1}(c)}$ denote its Levi-Civita connection. If I is an open interval and $f : I \rightarrow \mathbb{R}$ is a smooth, non-vanishing function, then the Robertson-Walker space $L_1^n(f, c)$ is defined as the Lorentzian warped product $I_1^1 \times_f R^{n-1}(c)$ whose metric tensor \tilde{g} is

$$\tilde{g} = \langle \cdot, \cdot \rangle = -dt^2 + f(t)^2 g_c,$$

where t is the natural coordinate in I and we use the notation $I_1^1 = (I, -dt^2)$. Note that if the warping function f is constant, then $L_1^n(f, c)$ becomes the Lorentzian product spaces $I_1^1 \times R^{n-1}(c)$. For any function f and $n = 4$, the Lorentzian manifolds $L_1^4(f, c)$ are known as Robertson-Walker space-times. Also, the vector field $\frac{\partial}{\partial t}$ is called the comoving observer field in general relativity, [12].

Let $\Pi_1 : I \times R^{n-1}(c) \rightarrow I$ and $\Pi_2 : I \times R^{n-1}(c) \rightarrow R^{n-1}(c)$ denote the canonical projections. For a given vector field X in $L_1^n(f, c)$, we define a function X_0 and a vector field \bar{X} by the orthogonal decomposition

$$X = X_0 \frac{\partial}{\partial t} + \bar{X},$$

that is,

$$X = X_0 \frac{\partial}{\partial t} + \sum_{i=1}^{n-1} X_i \frac{\partial}{\partial x_i} := (X_0, X_1, \dots, X_{n-1}),$$

where $(x_1, x_2, \dots, x_{n-1})$ is a Cartesian coordinate system in \mathbb{R}^{n-1} and we have

$$X_0 = -\left\langle \frac{\partial}{\partial t}, X \right\rangle, \quad \Pi_1^*(\bar{X}) = 0.$$

First, we would like to express the Levi-Civita connection of $L_1^n(f, c)$. Occasionally, by misusing terminology, we are going to put $(0, \bar{X}) = \bar{X}$.

We are going to use the following lemma which can be directly obtained from [10] (See also [4, Lemma 2.1]).

Lemma 1 *The Levi-Civita connection $\tilde{\nabla}$ of $L_1^n(f, c)$ is*

$$\tilde{\nabla}_X Y = \nabla_{\bar{X}}^0 Y + \frac{f'}{f} \left(\tilde{g}(\bar{X}, \bar{Y}) \frac{\partial}{\partial t} + X_0 \bar{Y} + Y_0 \bar{X} \right) \tag{1}$$

whenever X and Y are tangent to $L_1^n(f, c)$, where ∇^0 denotes the Levi-Civita connection of the Lorentzian Cartesian product space $L_1^n(1, c) = I_1^1 \times R^{n-1}(c)$.

Let the Riemann curvature tensor of $L_1^n(f, c)$ be denoted by \tilde{R} defined by

$$\tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X, Y]} Z$$

whenever X, Y, Z are tangent to $L_1^n(f, c)$. Note that the following lemma is obtained in [4].

Lemma 2 [4] *The Riemannian curvature tensor \tilde{R} of $L_1^n(f, c)$ satisfies*

$$\begin{aligned} \tilde{R}\left(\frac{\partial}{\partial t}, \bar{X}\right)\frac{\partial}{\partial t} &= \frac{f''}{f}\bar{X}, & \tilde{R}\left(\frac{\partial}{\partial t}, \bar{X}\right)\bar{Y} &= \frac{f''}{f}\langle \bar{X}, \bar{Y} \rangle \frac{\partial}{\partial t}, \\ \tilde{R}(\bar{X}, \bar{Y})\frac{\partial}{\partial t} &= 0, & \tilde{R}(\bar{X}, \bar{Y})\bar{Z} &= \frac{f'^2 + c}{f^2}(\langle \bar{Y}, \bar{Z} \rangle \bar{X} - \langle \bar{X}, \bar{Z} \rangle \bar{Y}) \end{aligned} \tag{2}$$

whenever $\Pi_1^*(\bar{X}) = \Pi_1^*(\bar{Y}) = \Pi_1^*(\bar{Z}) = 0$

2.2 Surfaces in Robertson-Walker Space-Times

Let M be a non-degenerated surface in $L_1^4(f, c)$, ∇ and g denote its Levi-Civita connection and metric tensor, respectively. One can define the second fundamental form h , the shape operator A , and the normal connection ∇^\perp of M by the Gauss and Weingarten formulas given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{3}$$

$$\tilde{\nabla}_X \xi = -A_\xi(X) + \nabla_X^\perp \xi, \tag{4}$$

where X and Y are tangent vectors to M and ξ is a normal vector to M . Note that the equation

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle \tag{5}$$

is satisfied.

On the other hand, one can define a tangent vector field T and a normal vector field η on M by the decomposition

$$\frac{\partial}{\partial t} \Big|_M = T + \eta, \tag{6}$$

(see [9, 11]).

The relative null space \mathcal{N}_p at a $p \in M$ is defined by

$$\mathcal{N}_p = \{X_p \in T_p M \mid h_p(X_p, Y_p) = 0 \text{ for all } Y_p \in T_p M\}.$$

If $\dim \mathcal{N}_p > 0$ for all $p \in M$, then M is said to have positive relative nullity, [4].

Remark 1 If $T = 0$ on a non-empty connected open subset \mathcal{O} of M , then \mathcal{O} is contained in a horizontal slice, i.e., $(\mathcal{O}, g) \subset \{t_0\} \times_{f(t_0)} R^3(c)$ for a $t_0 \in I$. In this case, by considering [4, Lemma 3.2], one can conclude that (\mathcal{O}, g) has positive relative nullity if and only if $f'(t_0) = 0$ and $\mathcal{O} = \{t_0\} \times_{f(t_0)} \bar{M}$, where \bar{M} is a surface in $R^3(c)$ with the Gaussian curvature $K = c$.

Remark 2 (Assumptions) Throughout this paper, we are going to assume the followings:

- (1) The space-time $L_1^4(f, c)$ has no open part with constant sectional curvature, i.e.,

$$\frac{f''(t)}{f(t)} - \frac{f'(t)^2 + c}{f(t)^2} \neq 0$$

for all $t \in I$.

- (2) T does not vanish on M (See Remark 1).
 (3) The relative null bundle \mathcal{N} is a smooth 1-dimensional distribution.
 (4) M is connected and it has no open part lying on a 3-dimensional totally geodesic hypersurface of $L_1^4(f, c)$.

On the other hand, by considering Lemma 2, one can observe that the Codazzi and Ricci equations turn into

$$\begin{aligned} ((X, X)Y_0 - \langle X, Y \rangle X_0) \left(-\frac{f''}{f} + \frac{(f')^2 + c}{f^2} \right) \eta \\ = (\nabla_X^\perp h)(Y, X) - (\nabla_Y^\perp h)(X, X), \end{aligned} \quad (7)$$

where $\nabla^\perp h$ is defined by

$$(\nabla_X^\perp h)(Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Moreover, the Ricci equation turns into

$$R^\perp(X, Y)\xi = h(X, A_\xi Y) - h(A_\xi X, Y), \quad (8)$$

where R^\perp denotes the normal curvature tensor of M defined by

$$R^\perp(X, Y)\xi = \nabla_X^\perp \nabla_Y^\perp \xi - \nabla_Y^\perp \nabla_X^\perp \xi - \nabla_{[X, Y]}^\perp \xi.$$

Consequently, M has flat normal bundle, i.e., $R^\perp = 0$ if and only if all shape operators of M have the same casual type, [4].

3 Surfaces with Positive Relative Nullity

3.1 Space-like Surfaces

Let M be a space-like surface in $L_1^4(f, c)$. In this case, by considering the decomposition (6), we define an orthonormal frame field $\{e_1, e_2; e_3, e_4\}$ and an ‘angle’ function θ by

$$\frac{\partial}{\partial t} \Big|_M = \sinh \theta e_1 + \cosh \theta e_3. \tag{9}$$

Note that we have $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = -\langle e_3, e_3 \rangle = \langle e_4, e_4 \rangle = 1$ because M is space-like.

Proposition 3 *Let M be a space-like surface in $L_1^4(f, c)$. Then, M has positive relative nullity if and only if its second fundamental form satisfies*

$$h(e_1, e_1) = h(e_1, e_2) = 0, \quad h(e_2, e_2) = \xi, \tag{10}$$

where e_1, e_2 are orthonormal tangent vector fields defined by (9) and ξ is a normal vector field.

Proof In order to prove the necessary condition, assume that M has positive relative nullity. Then, there exists an orthonormal base $\{\hat{e}_1, \hat{e}_2\}$ of the tangent bundle of M such that

$$h(\hat{e}_1, \hat{e}_1) = h(\hat{e}_1, \hat{e}_2) = 0, \quad h(\hat{e}_2, \hat{e}_2) = \xi \tag{11}$$

for a normal vector field ξ . By using the Codazzi equation (7) for $X = \hat{e}_1$ and $Y = \hat{e}_2$, we get

$$\left(-\frac{f''}{f} + \frac{(f')^2 + c}{f^2} \right) \beta \cosh \theta e_3 = -\langle \nabla_{\hat{e}_1} \hat{e}_1, \hat{e}_2 \rangle \xi, \tag{12}$$

where we put $\beta = (\hat{e}_2)_0$. Now, consider the open subset

$$\mathcal{O} = \{p \in M | \beta(p) \neq 0\}$$

of M . We are going to prove that \mathcal{O} is empty. Because of (12), we have $\xi = \alpha e_3$ on \mathcal{O} from which, along with (5), we get

$$A_{e_4} \Big|_{\mathcal{O}} = 0, \tag{13}$$

where α is a smooth function.

On the other hand, from (9) we get

$$\langle \tilde{\nabla}_X \frac{\partial}{\partial t}, e_4 \rangle = \langle \tilde{\nabla}_X (\sinh \theta e_1 + \cosh \theta e_3), e_4 \rangle \tag{14}$$

whenever X is a tangent vector field. By a further computation considering Lemma 1, (9) and (14), we obtain

$$\left\langle \left(\sinh \theta h(e_1, X) + \cosh \theta \nabla_X^\perp e_3 \right), e_4 \right\rangle = 0 \quad (15)$$

from which we get $\nabla^\perp e_3 = 0$ because of (13). Since the codimension of M is 2, we also have $\nabla^\perp e_4 = 0$. Therefore, (13) implies

$$\tilde{\nabla} e_4 = 0 \text{ on } \mathcal{O}.$$

Therefore, any component of \mathcal{O} lies on a totally geodesic hypersurface of $L_1^4(f, c)$. This result is in contradiction with the final assumption presented in Remark 2. Hence, \mathcal{O} is empty which yields that $\hat{e}_1 = \pm e_1$. This completes the proof of the necessary condition.

Converse is obvious because (10) implies $(e_1)_p \in \mathcal{N}_p$ for all $p \in M$. \square

Lemma 4 *Let M be a space-like surface in $L_1^4(f, c)$. Then, M has positive relative nullity if and only if the vector fields e_1, e_2, e_3, e_4 and the function θ defined by (9) satisfy*

$$\tilde{\nabla}_{e_1} e_1 = 0, \quad \tilde{\nabla}_{e_2} e_1 = \omega e_2, \quad (16a)$$

$$\tilde{\nabla}_{e_1} e_2 = 0, \quad \tilde{\nabla}_{e_2} e_2 = -\omega e_1 - h_{22}^3 e_3 + h_{22}^4 e_4, \quad (16b)$$

$$\tilde{\nabla}_{e_1} e_3 = 0, \quad \tilde{\nabla}_{e_2} e_3 = -h_{22}^3 e_2, \quad (16c)$$

$$\tilde{\nabla}_{e_1} e_4 = 0, \quad \tilde{\nabla}_{e_2} e_4 = -h_{22}^4 e_2, \quad (16d)$$

$$e_1(\theta) = \frac{f'}{f} \cosh \theta, \quad e_2(\theta) = 0 \quad (16e)$$

for some smooth functions ω , h_{22}^3 and h_{22}^4 .

Proof In order to prove the necessary condition, assume that M has positive relative nullity. Then, by Proposition 3, the equation (10) is satisfied. Then, we have (9) implies

$$A_{e_3} = \begin{pmatrix} 0 & 0 \\ 0 & h_{22}^3 \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & 0 \\ 0 & h_{22}^4 \end{pmatrix} \quad (17)$$

because of (5), where we put $h_{22}^\alpha = \langle \xi, e_\alpha \rangle$, $\alpha = 3, 4$. Note that (9) implies

$$\tilde{\nabla}_{e_1} \frac{\partial}{\partial t} = \tilde{\nabla}_{e_1} (\sinh \theta e_1 + \cosh \theta e_3), \quad (18)$$

$$\tilde{\nabla}_{e_2} \frac{\partial}{\partial t} = \tilde{\nabla}_{e_2} (\sinh \theta e_1 + \cosh \theta e_3).$$

By using Lemma 1, (10) and (17), we get

$$\begin{aligned} \frac{f'}{f}(\cosh^2 \theta e_1 + \sinh \theta \cosh \theta e_3) &= e_1(\theta)\zeta + \sinh \theta \nabla_{e_1} e_1 + \cosh \theta \nabla_{e_1}^\perp e_3, \\ \frac{f'}{f} e_2 &= e_2(\theta)\zeta + \sinh \theta \nabla_{e_2} e_1 + \cosh \theta (-h_{22}^3 e_2 + \nabla_{e_2}^\perp e_3), \end{aligned} \tag{19}$$

where $\zeta = \cosh \theta e_1 + \sinh \theta e_3$. By a further computation using (17) and (19), we obtain (16), for the function ω defined by

$$\omega = \langle \nabla_{e_2} e_1, e_2 \rangle.$$

This completes the proof of the necessary condition. Converse is obvious. □

Lemma 5 *Let M be a space-like surface in $L_1^4(f, c)$ with positive relative nullity. Then, there exists a constant a_0 such that*

$$\theta = \cosh^{-1} \frac{a_0}{f} \tag{20}$$

on M .

Proof Assume that M is a space-like surface in $L_1^4(f, c)$ with positive relative nullity. Then, Lemma 4 implies that (16) is satisfied. From (16a), (16b) and (16e) we have

$$\left[-\frac{1}{\sinh \theta} e_1, G e_2 \right] = 0$$

for a function G satisfying $e_1(G) - G\omega = 0$. Therefore, there exists a local coordinate system $(\mathcal{O}_p, (u, v))$ such that

$$e_1 = -\sinh \theta \partial_u = -\sinh \theta (1, \tilde{\phi}_u), \tag{21a}$$

$$e_2 = \frac{1}{G} \partial_v = \frac{1}{G} (0, \tilde{\phi}_v). \tag{21b}$$

On the other hand, because of (16e), on \mathcal{O}_p we have

$$\theta = \theta(u), \quad \theta'(u) \sinh \theta(u) + \frac{f'(u)}{f(u)} \cosh \theta(u) = 0$$

from which we get

$$\cosh \theta(u) = \frac{a_0}{f(u)}, \quad \sinh \theta(u) = \frac{\sqrt{a_0^2 - f(u)^2}}{f(u)}$$

for a constant a_0 on \mathcal{O}_p such that $a_0^2 - f(u)^2 > 0$. Since M is connected, the continuity of θ implies $\mathcal{O} = M$ and (20) on M . □

3.2 Time-like Surfaces

In this subsection, we are going to consider time-like surfaces with positive relative nullity. First, we obtain the analogous of Proposition 3 for time-like surfaces.

Proposition 6 *Let M be a time-like surface in $L_1^4(f, c)$. Then, M has positive relative nullity if and only if its second fundamental form satisfies (10), where e_1, e_2 form an orthonormal base for the tangent bundle of M such that e_1 is proportional to $\left(\frac{\partial}{\partial t}\right)^T$ and ξ is a normal vector field.*

Proof In order to prove the necessary condition, assume that M has positive relative nullity. Note that if $\eta = 0$ on M , then $e_1 = \frac{\partial}{\partial t}$ and e_2 satisfies

$$\tilde{\nabla}_{e_1} e_1 = 0 \text{ and } \tilde{\nabla}_{e_2} e_1 = \frac{f'}{f} e_2$$

because of Lemma 1 (See also [4, Lemma 3.3]). Therefore, the proof is completed for this case.

Now, we assume the existence of $p \in M$ such that $\eta(p) \neq 0$ and consider the open subset $\mathcal{O} = \{q \in M \mid \eta(q) \neq 0\}$. In this case, the vector fields T and η defined by (6) are time-like and space-like, respectively. First, we are going to prove that \mathcal{N}_p can not be degenerated. Towards contradiction, assume that \mathcal{N}_p is degenerated on \mathcal{O} . In this case, there exists a base field $\{X, Y\}$ of the tangent bundle of \mathcal{O} such that

$$\begin{aligned} h(X, X) = h(X, Y) = 0, \quad h(Y, Y) = \xi \\ \langle X, X \rangle = \langle Y, Y \rangle = 0, \quad \langle X, Y \rangle = -1. \end{aligned}$$

Consequently, the Codazzi equation (7) implies

$$X_0 \left(-\frac{f''}{f} + \frac{(f')^2 + c}{f^2} \right) \eta = 0$$

from which we obtain $X_0 = \langle X, T \rangle = 0$. However, this is a contradiction because X is light-like and T is time-like.

Consequently, one can define a local orthonormal frame field $\{e_1, e_2; e_3, e_4\}$ on \mathcal{O} and a function θ by

$$\frac{\partial}{\partial t} \Big|_{\mathcal{O}} = \cosh \theta e_1 + \sinh \theta e_3. \quad (22)$$

Similar to the proof of Proposition (3), we assume that there exists an orthonormal base $\{\hat{e}_1, \hat{e}_2\}$ of the tangent bundle of \mathcal{O} such that (11) is satisfied and $\langle \hat{e}_1, \hat{e}_1 \rangle = \varepsilon \in \{-1, 1\}$. By an analogous computation, we obtain $\hat{e}_1 = \pm e_1$ to complete the proof of the necessary condition. Converse is obvious. \square

Lemma 7 *Let M be a time-like surface in $L_1^4(f, c)$ with positive relative nullity. Then, there exists a constant a_0 such that*

$$\theta = \sinh^{-1} \frac{a_0}{f} \tag{23}$$

on M .

Proof If $\eta = 0$ on M , then (23) is satisfied for $a_0 = 0$. Assume the existence of $p \in M$ such that $\eta(p) \neq 0$. Similar to the proof of Proposition 6, we consider the connected component \mathcal{O} of $\{q \in M | \eta(q) \neq 0\}$ such that $p \in \mathcal{O}$. Let $\{e_1, e_2, e_3, e_4\}$ be the local orthonormal frame field on \mathcal{O} defined by (22).

Similar to the proof of Lemma 4, we consider the equation

$$\begin{aligned} \tilde{\nabla}_{e_1} \frac{\partial}{\partial t} &= \tilde{\nabla}_{e_1} (\cosh \theta e_1 + \sinh \theta e_3), \\ \tilde{\nabla}_{e_2} \frac{\partial}{\partial t} &= \tilde{\nabla}_{e_2} (\cosh \theta e_1 + \sinh \theta e_3), \end{aligned}$$

Lemma 1, (10) and (17), to get

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_2} e_1 = \omega e_2, \tag{24}$$

$$e_1(\theta) = -\frac{f'}{f} \sinh \theta, \quad e_2(\theta) = 0 \tag{25}$$

on \mathcal{O} , where ω is a smooth function.

Note that (24) and (25) imply $\left[\frac{1}{\cosh \theta} e_1, G e_2 \right] = 0$ if G is a smooth function satisfying

$$e_1(G) = G\omega. \tag{26}$$

Therefore, for any $q \in \mathcal{O}$ there exists a local coordinate system $(\mathcal{O}_q, (u, v))$ such that $\mathcal{O}_q \ni q$ and

$$e_1 = \cosh \theta \partial_u \quad e_2 = \frac{1}{G} \partial_v. \tag{27}$$

Consequently, (25) gives

$$\theta = \theta(u), \quad \theta'(u) \coth \theta(u) + \frac{f'(u)}{f(u)} = 0$$

on \mathcal{O}_q . Therefore, we have

$$\theta(u) = \sinh^{-1} \frac{a_0}{f(u)}$$

for a constant a_0 on \mathcal{O}_p . Since M is connected, the continuity of θ implies $\mathcal{O} = M$ and (23) on M . □

Consider the case $a_0 = 0$, i.e., $\eta = 0$ on M . Note that the classification of surfaces satisfying this condition is obtained in [4]:

Lemma 8 [4] *If M is a surface of $L_1^4(f, c)$ such that $\frac{\partial}{\partial t}$ is everywhere tangent to M , then M is an open part of $I \times_f \alpha \subset I \times_f \mathbb{R}^3(c)$, where $\alpha = \alpha(s)$ is a curve in $\mathbb{R}^3(c)$. Moreover, if $\alpha(s)$ is of unit speed, then the second fundamental form of M in $L_1^4(f, c)$ satisfies*

$$h(\partial_t, \partial_t) = h(\partial_t, \partial_s) = 0, \quad h(\partial_s, \partial_s) = \alpha''(s),$$

where we have identified the vector $\alpha''(s) := \nabla_{\alpha'(s)}^{R^3(c)} \alpha'$ in $\mathbb{R}^3(c)$ with its vertical lift in $L_1^4(f, c)$ via Π_2 .

As a direct consequence of this lemma, we have the following corollary.

Corollary 9 *If M is a time-like surface in $L_1^4(f, c)$ with $\eta = 0$, then M has positive relative nullity.*

Next, we are going to consider the case $a_0 \neq 0$. In this case, by considering the decomposition (6), we define an orthonormal frame field $\{e_1, e_2; e_3, e_4\}$ and an 'angle' function θ by

$$\frac{\partial}{\partial t} \Big|_M = \cosh \theta e_1 + \sinh \theta e_3. \quad (28)$$

Note that $\langle e_2, e_2 \rangle = -\langle e_1, e_1 \rangle = \langle e_3, e_3 \rangle = \langle e_4, e_4 \rangle = 1$ since M is time-like.

The proof of the following lemma is similar to the proof of Lemma 4.

Lemma 10 *Let M be a time-like surface in $L_1^4(f, c)$ with $\eta \neq 0$. Then, M has positive relative nullity if and only if the vector fields $e_1, e_2; e_3, e_4$ and the function θ defined by (28) satisfy*

$$\tilde{\nabla}_{e_1} e_1 = 0, \quad \tilde{\nabla}_{e_2} e_1 = \omega e_2, \quad (29a)$$

$$\tilde{\nabla}_{e_1} e_2 = 0, \quad \tilde{\nabla}_{e_2} e_2 = \omega e_1 + h_{22}^3 e_3 + h_{22}^4 e_4, \quad (29b)$$

$$\tilde{\nabla}_{e_1} e_3 = 0, \quad \tilde{\nabla}_{e_2} e_3 = -h_{22}^3 e_2, \quad (29c)$$

$$\tilde{\nabla}_{e_1} e_4 = 0, \quad \tilde{\nabla}_{e_2} e_4 = -h_{22}^4 e_2 \quad (29d)$$

for some smooth functions ω , h_{22}^3 and h_{22}^4 .

4 Surfaces in $L_1^4(f, 0)$

In this section, we are going to consider non-degenerated surfaces in $L_1^4(f, 0)$.

First, we construct a space-like surface with positive relative nullity.

Proposition 11 *Let $a_1, a_2, \phi_1 : I_v \rightarrow \mathbb{R}$ be some smooth functions and assume the existence of an $a_0 \in \mathbb{R}$ such that $a_0^2 - f(u)^2 > 0$ whenever $u \in I_u$, where $I_u \subset I$ and I_v are open intervals. Let ϕ_2, ϕ_3 and $\alpha_1, \alpha_2, \alpha_3 : I_v \rightarrow \mathbb{R}^3$ some smooth functions satisfying*

$$\phi_2' = -a_1 \phi_1, \quad \phi_3' = -a_2 \phi_1, \quad (30)$$

and

$$\begin{cases} \alpha_1' = a_1(v)\alpha_2 + a_2(v)\alpha_3, & \alpha_1(v_0) = C_1, \\ \alpha_2' = -a_1(v)\alpha_1, & \alpha_2(v_0) = C_2, \\ \alpha_3' = -a_2(v)\alpha_1, & \alpha_3(v_0) = C_3, \end{cases} \quad (31)$$

where $v_0 \in I_v$ and C_1, C_2, C_3 are constant orthonormal vectors in \mathbb{E}^3 .

Then, the space-like surface M in $L_1^4(f, 0)$ parametrized by

$$\begin{aligned} \phi(u, v) = & \left(u, \phi_1(v)\alpha_1(v) + \left(\int_{u_0}^u \frac{a_0}{f(\bar{u})\sqrt{a_0^2 - f(\bar{u})^2}} d\bar{u} + \phi_2(v) \right) \alpha_2(v) \right. \\ & \left. + \phi_3(v)\alpha_3(v) \right) \end{aligned} \tag{32}$$

has positive relative nullity.

Proof By a direct computation, we observe that the vector fields e_1, e_2, e_3, e_4 given by

$$\begin{aligned} e_1 &= -\frac{\sqrt{a_0^2 - f^2}}{f} \partial_u, & e_2 &= \frac{1}{\sqrt{G}} \partial_v, \\ e_3 &= \left(\frac{a_0}{f(u)}, \frac{\sqrt{a_0^2 - f(u)^2}}{f(u)} \alpha_2 \right), & e_4 &= \left(0, \frac{1}{f} \alpha_3 \right) \end{aligned}$$

form the orthonormal frame field defined by (9), where we put

$$G = f^2 \left(a_1 \left(\int_{u_0}^u \frac{a_0}{f(\bar{u})\sqrt{a_0^2 - f(\bar{u})^2}} d\bar{u} + \phi_2 \right) + a_2 \phi_3 - \phi_1' \right)^2.$$

By a further computation, we obtain that (16) is satisfied for

$$\begin{aligned} \omega &= \frac{f' \sqrt{G(a_0^2 - f^2)} + a_0 a_1 f}{\sqrt{G} f^2}, \\ h_{22}^3 &= \frac{a_1 f \sqrt{a_0^2 - f^2} + a_0 \sqrt{G} f'}{\sqrt{G} f^2}, \\ h_{22}^4 &= \frac{a_2}{\sqrt{G}}. \end{aligned} \tag{33}$$

Hence, M has positive relative nullity because of Lemma 4. □

Any two regular surfaces are said to be locally congruent if, for every point on one surface, there exists a neighborhood and an isometry of the ambient space mapping it onto a neighborhood of the other surface.

Now, we prove the following classification theorem:

Theorem 12 *Let M be a space-like surface in $L_1^4(f, 0)$. Then, M has positive relative nullity if and only if it is locally congruent to the surface described in Proposition 11.*

Proof The sufficient condition is proved in Proposition 11. In order to prove the necessary condition, we assume that M is a space-like surface in $L_1^4(f, 0)$ with positive relative nullity, $p \in M$ and let ϕ denote the position vector of M . Let $(\mathcal{O}_p, (u, v))$ be the local coordinate system constructed in the proof of Lemma 7. Also, we put $\tilde{\phi} = \Pi_2 \circ \phi$.

From (21a) and (21b), we have $\phi_u = (1, \tilde{\phi}_u)$ and $\phi_v = (0, \tilde{\phi}_v)$ which means that the position vector ϕ of M can be expressed as

$$\phi(u, v) = (u, \tilde{\phi}(u, v)) \quad (34)$$

on \mathcal{O}_p . Then, (21a) turns into

$$e_1 = -\frac{\sqrt{a_0^2 - f(u)^2}}{f(u)}(1, \tilde{\phi}_u). \quad (35)$$

Next, by considering (35), we use Lemma 1 for $c = 0$ to get

$$\tilde{\nabla}_{e_1} e_1 = \left(0, \frac{(a_0^2 - 2f^2) f' \tilde{\phi}_u + f(a_0^2 - f^2) \tilde{\phi}_{uu}}{f^3} \right). \quad (36)$$

By combining (36) with the first equation of (16a), we obtain

$$\tilde{\phi}_u(u, v) = \frac{a_0}{f(u)\sqrt{a_0^2 - f(u)^2}} \alpha_2(v) \quad (37)$$

for a smooth \mathbb{E}^3 -valued function α_2 .

On the other hand, we use Lemma 1 for $c = 0$, (21a) and (21b) to observe

$$\tilde{\nabla}_{e_1} e_2 = \left(0, \left(\left(\frac{1}{G} \right)_u + \frac{1}{G} \frac{f'}{f} \right) \tilde{\phi}_v + \frac{1}{G} \tilde{\phi}_{vu} \right). \quad (38)$$

Then, the first equation in (16b) implies

$$\tilde{\phi}_v(u, v) = \frac{G(u, v)}{f(u)} \alpha_1(v) \quad (39)$$

for a smooth \mathbb{E}^3 -valued function α_1 . Since $\{e_1, e_2\}$ is orthonormal, (21) implies that $\{\alpha_1, \alpha_2\}$ is orthonormal with respect to the metric g_0 .

Now, let α_3 be a \mathbb{E}^3 -valued function so that

$$g_0(\alpha_i, \alpha_j) = \delta_{ij}, \quad i, j = 1, 2, 3.$$

Then, we have

$$\tilde{\phi}(u, v) = \phi_1(u, v)\alpha_1(v) + \hat{\phi}_2(u, v)\alpha_2(v) + \phi_3(u, v)\alpha_3(v) \tag{40}$$

and

$$\begin{cases} \alpha'_1 = a_1(v)\alpha_2 + a_2(v)\alpha_3 \\ \alpha'_2 = -a_1(v)\alpha_1 + a_3(v)\alpha_3, \\ \alpha'_3 = -a_2(v)\alpha_1 - a_3(v)\alpha_2, \end{cases} \tag{41}$$

for some smooth functions $\phi_1, \hat{\phi}_2, \phi_3, a_1, a_2, a_3$. By a further computation using (37)-(41), we obtain $a_3 = 0$, (30) and

$$\partial_u \phi_1 = \partial_u \phi_3 = 0, \quad \hat{\phi}_2(u, v) = \int_{u_0}^u \frac{a_0}{f(\bar{u})\sqrt{a_0^2 - f(\bar{u})^2}} d\bar{u} + \phi_2(v). \tag{42}$$

By combining (40) and (42) with (34), we get (32). Moreover, because of $a_3 = 0$, (41) turns into (31). Hence, M is congruent to the surface described in Proposition 11. \square

In the remaining part, we deal with time-like surfaces in $L^4_1(f, 0)$. Next, we obtain analogues of Proposition 11 for time-like surfaces.

Proposition 13 *Let $a_1, a_2, \phi_1 : I_v \rightarrow \mathbb{R}$ be some smooth functions and assume that ϕ_2, ϕ_3 and $\alpha_1, \alpha_2, \alpha_3 : I_v \rightarrow \mathbb{R}^3$ are smooth functions satisfying (30) and (31) for a $v_0 \in I_v$ and some constant orthonormal vectors C_1, C_2, C_3 in \mathbb{E}^3 .*

Then, the time-like surface M in $L^4_1(f, 0)$ parametrized by

$$\begin{aligned} \phi(u, v) = & \left(u, \phi_1(v)\alpha_1(v) + \left(\int_{u_0}^u \frac{a_0}{f(\bar{u})\sqrt{a_0^2 + f(\bar{u})^2}} d\bar{u} + \phi_2(v) \right) \alpha_2(v) \right. \\ & \left. + \phi_3(v)\alpha_3(v) \right) \end{aligned} \tag{43}$$

has positive relative nullity, where a_0 is a non-zero constant.

Proof By a direct computation, we observe that M is time-like and (23) is satisfied for $a_0 \neq 0$. Similar to the proof of Proposition 11, we obtain that the vector fields e_1, e_2, e_3, e_4 given by

$$\begin{aligned} e_1 &= \frac{\sqrt{a_0^2 + f^2}}{f} \partial_u, & e_2 &= \frac{1}{\sqrt{G}} \partial_v, \\ e_3 &= \left(-\frac{a_0}{f(u)}, -\frac{\sqrt{a_0^2 + f(u)^2}}{f(u)} \alpha_2 \right), & e_4 &= \left(0, \frac{1}{f} \alpha_3 \right) \end{aligned}$$

form the orthonormal frame field defined by (28) satisfying (29) for

$$\begin{aligned}\omega &= \frac{f' \sqrt{G(a_0^2 + f^2)} + a_0 a_1 f}{\sqrt{G} f^2}, \\ h_{22}^3 &= \frac{a_1 f \sqrt{a_0^2 + f^2} + a_0 \sqrt{G} f'}{\sqrt{G} f^2}, \\ h_{22}^4 &= \frac{a_2}{\sqrt{G}},\end{aligned}\tag{44}$$

where we put

$$G = f^2 \left(a_1 \left(\int_{u_0}^u \frac{a_0}{f(z) \sqrt{a_0^2 + f(z)^2}} dz + \phi_2 \right) + a_2 \phi_3 - \phi_1' \right)^2.$$

Hence, M has positive relative nullity because of Lemma 10. \square

Next, we obtain the following classification theorem for time-like surfaces in $L_1^4(f, 0)$.

Theorem 14 *Let M be a time-like surface in $L_1^4(f, 0)$. Then, M has positive relative nullity if and only if it is one of the following surfaces:*

- (1) *The surface described in Lemma 8 for a curve α in \mathbb{E}^3 ,*
- (2) *The surface locally congruent to the surface described in Proposition 13.*

Proof Similar to the proof of Theorem 12, assume that M is a time-like surface in $L_1^4(f, 0)$ with positive relative nullity, $p \in M$ and let ϕ denote the position vector of M . By Lemma 7, (23) is satisfied for a constant a_0 . If $a_0 = 0$, then we have $\eta = 0$ on M and Lemma 8 implies that M is the surface given in the Case (1) of the theorem.

Now, we consider the case $a_0 \neq 0$. Then, Lemma 10 implies that (29) is satisfied. Let $(\mathcal{O}_p, (u, v))$ the coordinate system constructed in the proof of Lemma 7 satisfying (27) for a function G such that (26). Moreover, by Lemma 10, (29) is satisfied. Similar to the proof of Theorem 12, by considering the equations in (29a), we use Lemma 1 for $c = 0$ to get

$$\tilde{\phi}_u(u, v) = \frac{a_0}{f(u) \sqrt{a_0^2 + f(u)^2}} \alpha_2(v)\tag{45}$$

and (39) for some smooth \mathbb{E}^3 -valued functions α_1 and α_2 . Consequently, we get (40) on \mathcal{O}_p . By a further computation considering (39) and (45), we obtain that M is the surface given in Case (2) of the theorem. This completes the proof of the necessary condition.

The converse is proved in Lemma 8 and Proposition 13. \square

5 Surfaces in Lorentzian Product Spaces

In this section, we are going to consider the case $f = 1$ and $c = \pm 1$ to study surfaces in the product spaces $\mathbb{E}_1^1 \times R^3(c)$.

Let M be a non-degenerated surface in the Cartesian product space $L_1^4(1, c) = \mathbb{E}_1^1 \times R^3(c)$. In this section $\hat{\nabla}$ stands for the Levi-Civita connection of the flat ambient space \mathbb{E}_r^5 , where $r = 1$ if $c = 1$ and $r = 2$ if $c = -1$. Note that $\hat{\nabla}$ and $\tilde{\nabla}$ are related with

$$\hat{\nabla}_X Y = \tilde{\nabla}_X Y - c(\langle X, Y \rangle + \langle X, T \rangle \langle Y, T \rangle) e_5 \tag{46a}$$

$$\hat{\nabla}_X \xi = \tilde{\nabla}_X \xi - c \langle X, T \rangle \langle \xi, \eta \rangle e_5 \tag{46b}$$

whenever X, Y are tangent to M and ξ is a normal vector field, where T is the tangent vector field defined by (6) and e_5 denotes the restriction of the unit normal vector field of $\mathbb{E}_1^1 \times R^3(c)$ in \mathbb{E}_r^5 into M . Furthermore, we have

$$\hat{\nabla}_X e_5 = (X + \langle X, T \rangle T) + \langle X, T \rangle \eta. \tag{46c}$$

Now, assume that M has positive relative nullity. In this case, since $f = 1$, (16e) in Lemma 4 and Lemma 7 imply

$$\theta = \theta_0 \tag{47}$$

for a constant θ_0 , where θ is defined by (9) (resp. (28)) if M is space-like (resp. time-like). Note that we have $\theta_0 \neq 0$ when M is space-like.

5.1 Surfaces in $\mathbb{E}_1^1 \times \mathbb{S}^3$

In this subsection, we construct two families of surfaces in $\mathbb{E}_1^1 \times \mathbb{S}^3$ with positive relative nullity.

Proposition 15 *Let $a_1, a_2, a_3 : I_v \rightarrow \mathbb{R}$ be some smooth functions and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 : I_v \rightarrow \mathbb{R}^4$ are smooth functions satisfying*

$$\begin{cases} \alpha'_1 = a_1(v)\alpha_3, & \alpha_1(v_0) = C_1, \\ \alpha'_2 = a_2(v)\alpha_3, & \alpha_2(v_0) = C_2, \\ \alpha'_3 = -a_1(v)\alpha_1 - a_2(v)\alpha_2 + a_3(v)\alpha_4, & \alpha_3(v_0) = C_3, \\ \alpha'_4 = -a_3(v)\alpha_3, & \alpha_4(v_0) = C_4, \end{cases} \tag{48}$$

where $v_0 \in I_v$ and C_1, C_2, C_3, C_4 are constant orthonormal vectors in \mathbb{E}^4 .

Then, the space-like surface M in $\mathbb{E}_1^1 \times \mathbb{S}^3$ parametrized by

$$\phi(u, v) = (-u \sinh \theta_0, \cos(u \cosh \theta_0)\alpha_1(v) + \sin(u \cosh \theta_0)\alpha_2(v)) \tag{49}$$

has positive relative nullity, where θ_0 is a non-zero constant.

Proof By a direct computation, we obtain that (16) is satisfied for

$$\begin{aligned}\omega &= \frac{\cosh \theta_0(a_1 \sin(u \cosh \theta_0) - a_2 \cos(u \cosh \theta_0))}{a_1 \cos(u \cosh \theta_0) + a_2 \sin(u \cosh \theta_0)}, \\ h_{22}^3 &= \frac{\sinh \theta_0(a_2 \cos(u \cosh \theta_0) - a_1 \sin(u \cosh \theta_0))}{a_1 \cos(u \cosh \theta_0) + a_2 \sin(u \cosh \theta_0)}, \\ h_{22}^4 &= -\frac{a_3}{a_1 \cos(u \cosh \theta_0) + a_2 \sin(u \cosh \theta_0)}.\end{aligned}\quad (50)$$

Hence, M has positive relative nullity because of Lemma 4. \square

Proposition 16 Let $a_1, a_2, a_3 : I_v \rightarrow \mathbb{R}$ be some smooth functions and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 : I_v \rightarrow \mathbb{R}^4$ are smooth functions satisfying (48) for a $v_0 \in I_v$ and some constant orthonormal vectors C_1, C_2, C_3, C_4 in \mathbb{E}_1^4 .

Then, the time-like surface M in $\mathbb{E}_1^1 \times \mathbb{S}^3$ parametrized by

$$\phi(u, v) = (u \cosh \theta_0, \cos(u \sinh \theta_0)\alpha_1(v) + \sin(u \sinh \theta_0)\alpha_2(v)) \quad (51)$$

has positive relative nullity, where θ_0 is a constant.

Proof If $\theta_0 = 0$, then (51) turns into

$$\phi(u, v) = (u, \alpha_1(v)).$$

In this case, by Lemma 8, M has positive relative nullity.

Assume that $\theta_0 \neq 0$. By a direct computation, we obtain that (29) is satisfied for

$$\begin{aligned}\omega &= \frac{\sinh \theta_0(a_2 \cos(u \sinh \theta_0) - a_1 \sin(u \sinh \theta_0))}{a_1 \cos(u \sinh \theta_0) + a_2 \sin(u \sinh \theta_0)}, \\ h_{22}^3 &= \frac{\cosh \theta_0(a_2 \cos(u \sinh \theta_0) - a_1 \sin(u \sinh \theta_0))}{a_1 \cos(u \sinh \theta_0) + a_2 \sin(u \sinh \theta_0)}, \\ h_{22}^4 &= -\frac{a_3}{a_1 \cos(u \sinh \theta_0) + a_2 \sin(u \sinh \theta_0)}.\end{aligned}\quad (52)$$

Hence, M has positive relative nullity because of Lemma 10. \square

5.2 Surfaces in $\mathbb{E}_1^1 \times \mathbb{H}^3$

In this subsection, we construct two families of surfaces in $\mathbb{E}_1^1 \times \mathbb{H}^3$ with positive relative nullity.

Proposition 17 Let $a_1, a_2, a_3 : I_v \rightarrow \mathbb{R}$ be some smooth functions and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 : I_v \rightarrow \mathbb{R}^4$ are smooth functions satisfying

$$\begin{cases} \alpha_1' = a_1(v)\alpha_3, & \alpha_1(v_0) = C_1, \\ \alpha_2' = a_2(v)\alpha_3, & \alpha_2(v_0) = C_2, \\ \alpha_3' = a_1(v)\alpha_1 - a_2(v)\alpha_2 + a_3(v)\alpha_4, & \alpha_3(v_0) = C_3, \\ \alpha_4' = -a_3(v)\alpha_3, & \alpha_4(v_0) = C_4, \end{cases} \quad (53)$$

where $v_0 \in I_v$, C_1 is time-like, C_2, C_3, C_4 are space-like constant orthonormal vectors in \mathbb{E}_1^4 .

Then, the space-like surface M in $\mathbb{E}_1^4 \times \mathbb{H}^3$ parametrized by

$$\phi(u, v) = (-u \sinh \theta_0, \cosh(u \cosh \theta_0)\alpha_1(v) + \sinh(u \cosh \theta_0)\alpha_2(v)) \quad (54)$$

has positive relative nullity, where θ_0 is a non-zero constant.

Proof The surface M satisfies (16) for

$$\begin{aligned} \omega &= -\frac{\cosh \theta_0(a_1 \sinh(u \cosh \theta_0) + a_2 \cosh(u \cosh \theta_0))}{a_1 \cosh(u \cosh \theta_0) + a_2 \sinh(u \cosh \theta_0)}, \\ h_{22}^3 &= \frac{\sinh \theta_0(a_1 \sinh(u \cosh \theta_0) + a_2 \cosh(u \cosh \theta_0))}{a_1 \cosh(u \cosh \theta_0) + a_2 \sinh(u \cosh \theta_0)}, \\ h_{22}^4 &= -\frac{a_3}{a_1 \cosh(u \cosh \theta_0) + a_2 \sinh(u \cosh \theta_0)}. \end{aligned} \quad (55)$$

Hence, M has positive relative nullity because of Lemma 4. \square

Proposition 18 Let $a_1, a_2, a_3 : I_v \rightarrow \mathbb{R}$ be some smooth functions and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 : I_v \rightarrow \mathbb{R}^4$ are smooth functions satisfying (53) for a $v_0 \in I_v$ and some constant orthonormal vectors C_1, C_2, C_3, C_4 in \mathbb{E}_1^4 . Then, the time-like surface M in $\mathbb{E}_1^4 \times \mathbb{H}^3$ parametrized by

$$\phi(u, v) = (u \cosh \theta_0, \cosh(u \sinh \theta_0)\alpha_1(v) + \sinh(u \sinh \theta_0)\alpha_2(v)) \quad (56)$$

has positive relative nullity, where θ_0 is a non-zero constant.

Proof The surface M satisfies (29) for

$$\begin{aligned} \omega &= \frac{\sinh \theta_0(a_1 \sinh(u \sinh \theta_0) + a_2 \cosh(u \sinh \theta_0))}{a_1 \cosh(u \sinh \theta_0) + a_2 \sinh(u \sinh \theta_0)}, \\ h_{22}^3 &= \frac{\cosh \theta_0(a_1 \sinh(u \sinh \theta_0) + a_2 \cosh(u \sinh \theta_0))}{a_1 \cosh(u \sinh \theta_0) + a_2 \sinh(u \sinh \theta_0)}, \\ h_{22}^4 &= -\frac{a_3}{a_1 \cosh(u \sinh \theta_0) + a_2 \sinh(u \sinh \theta_0)}. \end{aligned} \quad (57)$$

Hence, M has positive relative nullity because of Lemma 4. \square

Remark 3 From the proofs of Proposition 15-Proposition 18, it can be easily seen that $h_{22}^4 = 0$ for $a_3 = 0$. In this case, the surface has open part lying on a 3-dimensional totally geodesic hypersurface of $L_1^4(f, c)$. Thus, the interior of

$$\{v_0 \in I_v | a_3(v_0) = 0\}$$

must be empty.

5.3 Local Classification Theorem

In this subsection, we obtain the following local classification of surfaces in $L_1^4(1, c)$ with positive relative nullity.

Theorem 19 *Let M be a non-degenerated surface in $L_1^4(1, c)$ with positive relative nullity. Then, the followings hold:*

- (i) *If M is a space-like surface and $c = 1$, then it is locally congruent to the surface described in Proposition 15.*
- (ii) *If M is a time-like surface and $c = 1$, then it is locally congruent to the surface described in Proposition 16.*
- (iii) *If M is a space-like surface and $c = -1$, then it is locally congruent to the surface described in Proposition 17.*
- (iv) *If M is a time-like surface and $c = -1$, then it is locally congruent to the surface described in Proposition 18.*

Proof First, we are going to prove (i) and (iii) of the theorem by considering a space-like surface M in $\mathbb{E}_1^1 \times R^3(c)$ with positive relative nullity. Let ϕ be the position vector of M . Then by combining the first equations in (16a), (16b) and (16d) with (46) we get

$$\hat{\nabla}_{e_1} e_1 = -c \left(\cosh^2 \theta_0 \right) e_5, \quad (58a)$$

$$\hat{\nabla}_{e_1} e_2 = 0, \quad (58b)$$

$$\hat{\nabla}_{e_1} e_4 = 0. \quad (58c)$$

On the other hand, by combining (47) with (9), we obtain

$$\frac{\partial}{\partial t} \Big|_M = \sinh \theta_0 e_1 + \cosh \theta_0 e_3. \quad (59)$$

Now, let $p \in M$. Similar to the proof of Theorem 12, by considering $[e_1, Ge_2] = (e_1(G) - G\omega)e_2$, we choose a local coordinate system $(\mathcal{O}_p, (u, v))$ such that

$$e_1 = \partial_u = (-\sinh \theta_0, \tilde{\phi}_u), \quad (60a)$$

$$e_2 = \frac{1}{G} \partial_v = \frac{1}{G} (0, \tilde{\phi}_v), \quad (60b)$$

where G is a smooth function satisfying $e_1(G) - G\omega = 0$ and $p \in \mathcal{O}_p$. Note that because of (60), up to a suitable translation on u , we have

$$\phi(u, v) = (-u \sinh \theta_0, \tilde{\phi}(u, v)) \quad (61)$$

on \mathcal{O}_p , where $\tilde{\phi} = \Pi_2(\phi)$.

Next, by a direct computation considering (58), (60) and (61), we get

$$\tilde{\phi}_{uu} + c \cosh^2 \theta_0 \tilde{\phi} = 0, \tag{62a}$$

$$e_2 = (0, \alpha_3(v)), \tag{62b}$$

$$e_4 = (0, \alpha_4(v)), \tag{62c}$$

where $\alpha_3, \alpha_4 : I_v \rightarrow \mathbb{R}^4$ are some smooth functions and I_v is an open interval. By solving (62a), we get

$$\tilde{\phi}(u, v) = \cos(u \cosh \theta_0) \alpha_1(v) + \sin(u \cosh \theta_0) \alpha_2(v) \tag{63}$$

for $c = 1$ and

$$\tilde{\phi}(u, v) = \cosh(u \cosh \theta_0) \alpha_1(v) + \sinh(u \cosh \theta_0) \alpha_2(v) \tag{64}$$

for $c = -1$, where $\alpha_1, \alpha_2 : I_v \rightarrow \mathbb{R}^4$ are some smooth functions. By combining (61) with (63) and (64), we get (49) and (54), respectively. Moreover, since $\{e_1, e_2, e_4\}$ is an orthonormal set, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ must be orthonormal in \mathbb{E}_{r-1}^4 . By a further computation we obtain (48) (resp. (53)) for $c = 1$ (resp. $c = -1$). Hence we have (i) and (iii) of the theorem. (ii) and (iv) is proved by a similar technique. \square

6 Conclusion

In this work, we investigated the space-like and time-like surfaces with positive relative nullity in Robertson-Walker space-times $L_1^4(f, c)$. Without assuming the marginally trapped condition, we established necessary and sufficient conditions for such surfaces and calculated the corresponding Levi-Civita connections. In the case $c = 0$ where the ambient space is the warped product $L_1^4(f, 0) = I_1^1 \times_f \mathbb{E}^3$ we obtained local classification theorems for space-like and time-like surfaces with positive relative nullity. These results provide explicit parametrizations and characterize such surfaces up to local congruence.

In the another special case where the warping function $f(t)$ is constant, we provided a complete local classification of such surfaces in the Lorentzian product spaces $\mathbb{E}_1^1 \times \mathbb{S}^3$ and $\mathbb{E}_1^1 \times \mathbb{H}^3$.

The concept of positive relative nullity is a geometrically meaningful condition in the study of pseudo-Riemannian submanifolds, as it highlights directions along which the second fundamental form vanishes and can lead to simplifications in certain curvature computations. While most previous studies have focused on marginally trapped surfaces in specific Lorentzian space-forms, this study removes this restriction and addresses a broader class of surfaces. Also, the ambient space is considered as Robertson-Walker space-time which is a general family of Lorentzian manifolds that encompasses physically significant models such as Minkowski, de Sitter, anti-de Sitter, and Friedmann cosmological models. Thus, these findings generalize previous results in the literature, and contribute to a deeper understanding in Lorentzian warped product spaces.

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