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# Central Armendariz Rings

[1]NAZIM AGAYEV, [2]GONCA GÜNGÖROĞLU,
[3]ABDULLAH HARMANCI AND [4] SAIT HALICIOĞLU
[1]Department of Computer Engineering, University of Lefke, Cyprus
[2]Department of Mathematics, Adnan Menderes University, Aydın, Turkey
[3]Department of Mathematics, Hacettepe University, Ankara, Turkey
[4] Department of Mathematics, Ankara University, Ankara, Turkey
[1]agayev@eul.edu.tr, [2]gungoroglu@adu.edu.tr,
[3]harmanci@hacettepe.edu.tr, [4] halici@science.ankara.edu.tr

**Abstract.** We introduce the notion of central Armendariz rings which are a generalization of Armendariz rings and investigate their properties. We show that the class of central Armendariz rings lies strictly between classes of Armendariz rings and abelian rings. For a ring $R$, we prove that $R$ is central Armendariz if and only if the polynomial ring $R[x]$ is central Armendariz if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is central Armendariz. Moreover, it is proven that if $R$ is reduced, then $R[x]/(x^n)$ is central Armendariz, the converse holds if $R$ is semiprime, where $(x^n)$ is the ideal generated by $x^n$ and $n \geq 2$. Among others we also show that $R$ is a reduced ring if and only if the matrix ring $T_n^{n-2}(R)$ is central Armendariz, for a natural number $n \geq 3$ and $k = [n/2]$.

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## 1. Introduction

In [16], Rege and Chhawchharia introduced the notion of an Armendariz ring. A ring $R$ is called *Armendariz* if for any $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{s} b_j x^j \in R[x]$, $f(x)g(x) = 0$ implies that $a_i b_j = 0$ for all $i$ and $j$. The name of the ring was given due to Armendariz who proved that reduced rings (i.e., rings without nonzero nilpotent elements) satisfied this condition [3]. A number of papers have been written on the Armendariz property of rings (see, e.g., [2, 6, 10, 12, 13]). So far, Armendariz rings are generalized in different ways. Let $\alpha$ be an endomorphism of a ring $R$. Hong, Kwak and Rizvi [7] give a possible generalization of Armendariz rings. A ring $R$ is called $\alpha$-*Armendariz* if for any $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{s} b_j x^j \in R[x; \alpha]$, $f(x)g(x) = 0$ implies $a_i b_j = 0$ for all $i$ and $j$. According to Hong, Kim and Kwak

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[6], a ring $R$ is called *$\alpha$-skew Armendariz* if $f(x)g(x) = 0$, where $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{s} b_j x^j \in R[x;\alpha]$, implies $a_i\alpha^i(b_j) = 0$ for each $i$ and $j$. Lee and Zhou defined $\alpha$-Armendariz module in [14] so that Armendariz rings are generalized to modules as $\alpha$-Armendariz modules, and so $\alpha$-Armendariz rings. A ring $R$ is called *$\alpha$-Armendariz* if

(1) for any $a, b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$,
(2) for any $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{s} b_j x^j \in R[x;\alpha]$, $f(x)g(x) = 0$ implies $a_i\alpha^i(b_j) = 0$ for all $i$ and $j$.

In [15], Liu and Zhao define and investigate weak Armendariz rings. A ring $R$ is called *weak Armendariz* if whenever $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{s} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j$ is a nilpotent element of $R$ for each $i$ and $j$.

In this paper, we call a ring $R$ *central Armendariz* if whenever $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{s} b_j x^j \in R[x]$, $f(x)g(x) = 0$ implies $a_ib_j$ is a central element of $R$ for each $i$ and $j$. Clearly, Armendariz rings are central Armendariz. We supply some examples to show that all central Armendariz rings need not be Armendariz. Among others we prove that central Armendariz rings are abelian rings and there exists an abelian ring but not central Armendariz. Therefore the class of central Armendariz rings lies strictly between classes of Armendariz rings and abelian rings. It is shown that a ring $R$ is central Armendariz if and only if the polynomial ring $R[x]$ is central Armendariz if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is central Armendariz. It is proven that if $R$ is reduced, then $R[x]/(x^n)$ is central Armendariz, the converse holds if $R$ is semiprime, where $(x^n)$ is the ideal generated by $x^n$ and $n \geq 2$. In [13], Lee and Zhou prove that, if $R$ is reduced ring, then $T_n^k(R)$ is Armendariz ring for $k = [n/2]$. We prove that the converse statement is also true and we show that $R$ is a reduced ring if and only if the matrix ring $T_n^{n-2}(R)$ is central Armendariz, for $n \geq 3$ and $k = [n/2]$. For an ideal $I$ of $R$, if $R/I$ is central Armendariz and $I$ is reduced, then $R$ is central Armendariz.

Throughout this paper, $R$ denotes an associative ring with identity unless specifed otherwise. The center of a ring $R$ will be denoted by $C(R)$. For a positive integer $n$, $\mathbb{Z}_n$ denotes the ring of integers $\mathbb{Z}$ modulo $n$. We write $R[x], R[[x]], R[x, x^{-1}]$ and $R[[x, x^{-1}]]$ for the polynomial ring, the power series ring, the Laurent polynomial ring and the Laurent power series ring over a ring $R$, respectively.

## 2. Central Armendariz rings

In this section, central Armendariz rings are introduced as a generalization of Armendariz rings. A ring $R$ is called *central Armendariz* if for any $f(x) = \sum_{i=0}^{s} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$, $f(x)g(x) = 0$ implies that $a_ib_j \in C(R)$ for all $i$ and $j$. Note that all commutative rings, reduced rings, Armendariz rings and subrings of central Armendariz rings are central Armendariz.

Let $R$ be a ring and let $M$ be an $(R, R)$-bimodule. The *trivial extension* of $R$ by $M$ is defined to be the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$.

One may suspect that if $R$ is a central Armendariz ring, then $R$ is Armendariz. But this is not the case.

**Example 2.1.** There exist central Armendariz rings but not Armendariz.

*Proof.* (1) Let $\mathbb{Z}_3[x;y]$ be the polynomial ring over $\mathbb{Z}_3$ in commuting indeterminates $x$ and $y$. Consider the ring $R = \mathbb{Z}_3[x;y]/(x^3;x^2y^2;y^3)$. The commutativity of $R$ implies that it is central Armendariz. By [12, Example 3.2], $R$ is not Armendariz.

(2) In [12, Theorem 2.3], Lee and Wong proved that $R$ is a reduced ring if and only if $T(R,R)$ is Armendariz. Hence, if $n$ is non-square-free, then $T(\mathbb{Z}_n,\mathbb{Z}_n)$ is not Armendariz. However, since $Z_n$ is commutative, $T(\mathbb{Z}_n,\mathbb{Z}_n)$ is also commutative, so it is central Armendariz. ∎

Recall that a ring $R$ is said to be *abelian* if every idempotent of it belong to the $C(R)$.

**Proposition 2.1.** *For a ring $R$ the following are equivalent:*

(1) *$R$ is central Armendariz.*
(2) *$R$ is abelian, $eR$ and $(1-e)R$ are central Armendariz for any idempotent $e \in R$.*
(3) *There is a central idempotent $e \in R$ with $eR$ and $(1-e)R$ are central Armendariz.*

*Proof.* (1) $\Longrightarrow$ (2). Since subrings of central Armendariz rings are central Armendariz, we prove only $R$ is abelian. Let $e$ be any idempotent in $R$. Consider $f(x) = e - er(1-e)x, g(x) = (1-e) + er(1-e)x \in R[x]$ for any $r \in R$. Then $f(x)g(x) = 0$. By hypothesis $er(1-e)$ is central and so $er(1-e) = 0$. Hence $er = ere$ for all $r \in R$. Similarly, consider $h(x) = (1-e)-(1-e)rex$ and $t(x) = e+(1-e)rex$ in $R[x]$ for any $r \in R$. Then $h(x)t(x) = 0$. As before $(1-e)re = 0$ and $ere = re$ for all $r \in R$. It follows that $e$ is central element of $R$, that is, $R$ is abelian.
(2) $\Longrightarrow$ (3). Clear.
(3) $\Longrightarrow$ (1). Let $e$ be a central idempotent of $R$, $f(x) = \sum_{j=0}^{n} a_j x^j$ and $g(x) = \sum_{j=0}^{t} b_j x^j$ nonzero polynomials in $R[x]$. Assume that $f(x)g(x) = 0$. Let $f_1 = ef(x)$, $f_2 = (1-e)f(x)$, $g_1 = eg(x)$, $g_2 = (1-e)g(x)$. Then $f_1(x)g_1(x) = 0$ in $(eR)[x]$ and $f_2(x)g_2(x) = 0$ in $((1-e)R)[x]$. By (2) $ea_ieb_j$ is central in $eR$ and $(1-e)a_i(1-e)b_j$ is central in $(1-e)R$. Since $e$ and $1-e$ central in $R$, $R = eR \oplus (1-e)R$ and so $a_ib_j = ea_ib_j + (1-e)a_ib_j$ is central in $R$ for all $0 \le i \le n$, $0 \le j \le t$. Then $R$ is central Armendariz. ∎

**Corollary 2.1.** [10, Lemma 7] *Armendariz rings are abelian.*

*Proof.* Clear from Proposition 2.1. ∎

The next example shows that abelian rings need not be central Armendariz in general.

**Example 2.2.** There exists an abelian ring but not central Armendariz.

*Proof.* Let $\mathbb{Z}^{2\times 2}$ be the $2 \times 2$ full matrix ring over $\mathbb{Z}$ and consider the ring

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} \mid a \equiv d (\text{mod } 2), b \equiv c \equiv 0 (\text{mod } 2) \right\}$$

The only idempotents in $R$ are $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so $R$ is an abelian ring.

Let
$f(x) = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} x,\ g(x) = \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} x \in R[x]$. Then $f(x)g(x) = 0$, but $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}$ is not central in $R$. Therefore $R$ is not central Armendariz. ∎

In [9], *Baer rings* are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. According to Clark [5], a ring is said to be *quasi-Baer* if the right annihilator of each right ideal of $R$ is generated (as a right ideal) by an idempotent. These definitions are left-right symmetric. A ring $R$ is called *right principally quasi-Baer* (*or simply, right p.q.-Baer*) [4] if the right annihilator of a principal right ideal of $R$ is generated by an idempotent. Finally, a ring $R$ is called *right principal projective* (it or simply, right p.p.-ring) if the right annihilator of an element of $R$ is generated by an idempotent.

Clearly, any Armendariz ring is central Armendariz. In the following, we show that the converse holds if the ring is a right p.p.-ring.

**Theorem 2.1.** *If the ring $R$ is Armendariz, then $R$ is central Armendariz. The converse holds if $R$ is a right p.p.-ring.*

*Proof.* Suppose $R$ is a central Armendariz and right p.p.-ring. By Proposition 2.1, $R$ is abelian. Let $f(x) = \sum_{i=0}^{s} a_i x^i, g(x) = \sum_{j=0}^{t} b_j x^j \in R[x]$. Assume that $f(x)g(x) = 0$. Then we have:

$$a_0 b_0 = 0 \tag{2.1}$$

$$a_0 b_1 + a_1 b_0 = 0 \tag{2.2}$$

$$a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \tag{2.3}$$

$$\dots$$

By hypothesis there exist idempotents $e_i \in R$ such that $r(a_i) = e_i R$ for all $i$. So $b_0 = e_0 b_0$ and $a_0 e_0 = 0$. Multiplying (2.2) by $e_0$ from the right, we have $0 = a_0 b_1 e_0 + a_1 b_0 e_0 = a_0 e_0 b_1 + a_1 b_0 e_0 = a_1 b_0$. By (2.2) $a_0 b_1 = 0$ and so $b_1 = e_0 b_1$. Again, multiplying (2.3) by $e_0$ from the right, we have $0 = a_0 b_2 e_0 + a_1 b_1 e_0 + a_2 b_0 e_0 = a_1 b_1 + a_2 b_0$. Multiplying this equation by $e_1$ from the right, we have $0 = a_1 b_1 e_1 + a_2 b_0 e_1 = a_2 b_0$. Continuing this process, we have $a_i b_j = 0$ for all $1 \le i \le s$ and $1 \le j \le t$. Hence $R$ is Armendariz. This completes the proof. ∎

The next example shows that the assumption of “right p.p.-ring” in Theorem 2.1 is not superfluous.

**Example 2.3.** There exists a central Armendariz ring which is neither a right p.p.-ring nor an Armendariz ring.

*Proof.* Since $R = T(\mathbb{Z}_8, \mathbb{Z}_8)$ is commutative, $R$ is central Armendariz. But by [16, Example 3.2], $R$ is not Armendariz. Moreover, since the principal ideal $I = \begin{pmatrix} 0 & \mathbb{Z}_8 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R$ is not projective, $R$ is not a right p.p.-ring. ∎

In [2], it is proven that a ring $R$ is Armendariz if and only if its polynomial ring $R[x]$ is Armendariz. Next theorem shows that this is also true for central Armendariz rings.

**Theorem 2.2.** *A ring $R$ is central Armendariz ring if and only if $R[x]$ is a central Armendariz ring.*

*Proof.* One way is evident since any subring of a central Armendariz ring is again central Armendariz. Conversely, suppose that $R$ is central Armendariz and let $f(y) = f_0+f_1y+\ldots+f_ny^n, g(y) = g_0+g_1y+\ldots+g_my^m \in R[x][y]$ with $f(y)g(y) = 0$, where $f_i = a_{i0}+a_{i1}x+\ldots+a_{in_i}x^{n_i}, g_j = b_{j0}+b_{j1}x+\ldots+b_{jm_j}x^{m_j} \in R[x]$. We prove that each $f_ig_j \in C(R[x])$. Let $t = \deg\ f_0+\ldots+\deg\ f_n+\deg\ g_0+\ldots+\deg\ g_m$ where the degree is as polynomials in $x$ and the degree of the zero polynomial is taken to be zero. Then $f(x^t) = f_0+f_1x^t+\ldots+f_nx^{tn}, g(x^t) = g_0+g_1x^t+\ldots+g_mx^{tm} \in R[x]$ and the set of coefficients of the $f_i$'s (resp. $g_j$'s) equals the set of coefficients of the $f(x^t)$ (resp. $g(x^t)$). Since $f(y)g(y) = 0$ and $x$ commutes with elements of $R$, $f(x^t)g(x^t) = 0$. Since $R$ is central Armendariz, $a_{is_i}b_{jr_j} \in C(R)$, where $0 \le s_i \le n_i, 0 \le r_j \le m_j$. Since $C(R)$ is closed under addition, $f_ig_j \in C(R[x])$. ∎

If $R$ is a reduced ring, by [16, Proposition 2.5] the trivial extension $T(R,R)$ is Armendariz and so it is central Armendariz. One may ask that if $T(R,R)$ is a central Armendariz ring, then $R$ is a reduced ring. But this is not the case. For non-square-free positive integer $n$, $T(\mathbb{Z}_n,\mathbb{Z}_n)$ is central Armendariz, but $\mathbb{Z}_n$ is not a reduced ring. For more general case, we have

**Theorem 2.3.** *Let $R$ be a ring. If $R$ is reduced, then $R[x]/(x^n)$ is central Armendariz, for a natural number $n \ge 2$. The converse holds if $R$ is semiprime.*

*Proof.* If $R$ is reduced, then $R[x]/(x^n)$ is Armendariz by [2, Theorem 5]. Hence by Theorem 2.1, $R[x]/(x^n)$ is central Armendariz. Conversely, suppose $R[x]/(x^n)$ is central Armendariz and $R$ is a semiprime ring. Let $r \in R$ with $r^k = 0$. Since $r$ and $\overline{x} = x + (x^n)$ commute,

$$0 = (r - \overline{x}y)(r^{k-1} + r^{k-2}\overline{x}y + \ldots + (\overline{x})^{k-1}y^{k-1})$$

where $y$ is indeterminate. Since $R[x]/(x^n)$ is central Armendariz, $r^{k-1}\overline{x} \in C(R[x]/(x^n))$. So $r^{k-1}\overline{x}\,\overline{a} = \overline{a}r^{k-1}\overline{x}$ for any $a \in R$, then $r^{k-1}a = ar^{k-1}$, hence $(r^{k-1}R)^n = 0$. Since $R$ is semiprime, $r^{k-1} = 0$. This completes the proof. ∎

It is clear that every reduced ring $R$ is reversible, i.e., for any $a$, $b \in R$, $ab = 0$ implies $ba = 0$. In Theorem 2.3, it can be asked that the reducibility of $R$ may be replaced by the reversibility of $R$. But the following example erases the possibility.

**Example 2.4.** There exists a reversible ring $R$ such that the trivial extension $T(R,R)$ of $R$ is not central Armendariz.

*Proof.* Let $\mathbb{H}$ be the Hamiltonian quaternions over the real number field and $R = T(\mathbb{H},\mathbb{H})$. By [11, Proposition 1.6], $R$ is reversible. Let $S$ be the trivial extension of $R$ by $R$. Consider the polynomials $f(x) = A + Bx$, $g(x) = C + Dx \in S[x]$, where

$$A = \begin{pmatrix} \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \end{pmatrix}, \quad B = \begin{pmatrix} \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$C = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}, \quad D = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -j & 0 \\ 0 & -j \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

Then $f(x)g(x) = 0$. However

$$AD = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & j-k \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

and for

$$E = \begin{pmatrix} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \end{pmatrix},$$

we have $ADE \neq EAD$. Therefore $S$ is not central Armendariz. ∎

Let $R$ be any ring. For any integer $n \geq 2$, consider the ring $R^{n\times n}$ of $n \times n$ matrices and the ring $T_n(R)$ of $n \times n$ upper triangular matrices over $R$. The rings $R^{n\times n}$ and $T_n(R)$ contain non-central idempotents. Therefore they are not abelian. By Proposition 2.1, these rings are not central Armendariz.

Now we introduce a notation for some subrings of $T_n(R)$ that will be central Armendariz. Let $k$ be a natural number smaller than $n$. Say

$$T_n^k(R) = \left\{ \sum_{i=j}^{n} \sum_{j=1}^{k} a_j e_{(i-j+1)i} + \sum_{i=j}^{n-k} \sum_{j=1}^{n-k} r_{ij} e_{j(k+i)} : a_j, r_{ij} \in R \right\}$$

where $e_{ij}$ are matrix units. Elements of $T_n^k(R)$ are in the form

$$\begin{bmatrix} x_1 & x_2 & \dots & x_k & a_{1(k+1)} & a_{1(k+2)} & \dots & a_{1n} \\ 0 & x_1 & \dots & x_{k-1} & x_k & a_{2(k+2)} & \dots & a_{2n} \\ 0 & 0 & x_1 & \dots & & & & a_{3n} \\ & & & \dots & & & & \\ & & & & & & & x_1 \end{bmatrix}$$

where $x_i, a_{js} \in R$, $1 \leq i \leq k$, $1 \leq j \leq n-k$ and $k+1 \leq s \leq n$.

For a reduced ring $R$, our aim is to investigate necessary and sufficient conditions for $T_n^k(R)$ to be central Armendariz. In [13], Lee and Zhou prove that, if $R$ is a reduced ring, then $T_n^k(R)$ is an Armendariz ring for $k = [n/2]$. In Theorem 2.5, we show that the converse of this result is also true. Moreover, we also prove that $R$ is a reduced ring if and only if $T_n^{n-2}(R)$ is a central Armendariz ring. To prove this property, we need the following lemma.

**Lemma 2.1.** *Let $R$ be a ring. Suppose that there exist $a, b \in R$ such that $a^2 = b^2 = 0$ and $ab = ba$ is not central. Then $R$ is not a central Armendariz ring.*

*Proof.* $(a + bx)(a - bx) = 0$ in $R[x]$, but $ab$ is not central. So, $R$ is not a central Armendariz ring. ∎

**Theorem 2.4.** *Let $n \geq 3$ be a natural number. Then $R$ is a reduced ring if and only if $T_n^k(R)$ is a central Armendariz ring, where $[n/2] \leq k \leq n-2$.*

*Proof.* Let $R$ be a reduced ring. In [13], it is shown that $T_n^k(R)$ is an Armendariz ring and so it is central Armendariz. Conversely, suppose that $R$ is not a reduced ring. Choose a nonzero element $a \in R$ with square zero. Then for elements $A = a(e_{11} + e_{22} + \ldots + e_{nn})$, $B = e_{1(k+1)} + e_{1(k+2)} + \ldots + e_{1n}$ in $T_n^k(R)$, $A^2 = B^2 = 0$ and $AB = BA$ is not central, since $(AB)(e_{1(n-k)} + e_{2(n-k+1)} + \ldots + e_{k(n-1)} + e_{(k+1)n}) = ae_{1n} \neq 0$. Therefore, from Lemma 2.1, $T_n^k(R)$ is not a central Armendariz ring. This completes the proof. ∎

**Theorem 2.5.** *Let $R$ be a ring, $n \geq 3$ be a natural number and $k = [n/2]$. Then the following are equivalent:*

(1) *$R$ is a reduced ring.*
(2) *$T_n^k(R)$ is an Armendariz ring.*
(3) *$T_n^{n-2}(R)$ is a central Armendariz ring.*

*Proof.* (1) $\Longrightarrow$ (2). See [13].
(2) $\Longrightarrow$ (3). It is evident since subrings of Armendariz rings are Armendariz.
(3) $\Longrightarrow$ (1). Clear from Theorem 2.4. ∎

Note that the homomorphic image of a central Armendariz ring need not be central Armendariz. If the ring $R$ is commutative or Gaussian, then by [2, Theorem 8], homomorphic image of $R$ is Armendariz, therefore it is central Armendariz. Note the following result.

**Theorem 2.6.** *Let $I$ be an ideal of a ring $R$. If $I$ is reduced as a ring and $R/I$ is a central Armendariz ring, then $R$ is central Armendariz.*

*Proof.* Let $a, b \in R$. If $ab = 0$, then $(bIa)^2 = 0$. Since $bIa \subseteq I$ and $I$ is reduced, $bIa = 0$. Also, $(aIb)^3 \subseteq (aIb)(I)(aIb) = 0$. Therefore $aIb = 0$. Assume $f(x) = a_0 + \ldots + a_n x^n, g(x) = b_0 + \ldots + b_m x^m \in R[x]$ and $f(x)g(x) = 0$. Then

$$a_0 b_0 = 0 \tag{2.4}$$

$$a_0 b_1 + a_1 b_0 = 0 \tag{2.5}$$

$$a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \tag{2.6}$$

$$\ldots$$

We first show that for any $a_i b_j$, $a_i I b_j = b_j I a_i = 0$. Multiply (2.5) from the right by $Ib_0$, we have $a_1 b_0 I b_0 = 0$, since $a_0 b_1 I b_0 = 0$. Then $(b_0 I a_1)^3 \subseteq b_0 I(a_1 b_0 I a_1 b_0) I a_1 = 0$. Hence $b_0 I a_1 = 0$. This implies $a_1 I b_0 = 0$. Multiply (2.5) from the left by $a_0 I$, we have $a_0 I a_0 b_1 + a_0 I a_1 b_0 = 0$ and so $a_0 I a_0 b_1 = 0$. Thus $(b_1 I a_0)^3 = 0$ and $b_1 I a_0 = 0$. Therefore $a_0 I b_1 = 0$. Now multiply (2.6) from right by $Ib_0$. Then $a_2 b_0 I b_0 = 0$ and $(b_0 I a_2)^3 = 0$. So, $b_0 I a_2 = 0$ and $a_2 I b_0 = 0$. Now from (2.6) we have $a_0 b_2 I + a_1 b_1 I + a_2 b_0 I = 0$. Since square of $a_0 b_2 I$ and $a_2 b_0 I$ are zero, $a_0 b_2 I = a_2 b_0 I = 0$. So $a_1 b_1 I = 0$. Then $(b_1 I a_1)^2 = 0$ and $b_1 I a_1 = 0$. So $a_1 I b_1 = 0$. Continuing in this way we have $a_i I b_j = b_j I a_i = 0$.

Since $R/I$ is central Armendariz, it follows that $\overline{a_i}\overline{b_j} \in C(R/I)$. So $a_ib_jr - ra_ib_j \in I$ for any $r \in R$. Now from the above results, it can be easily seen that $(a_ib_jr - ra_ib_j)I(a_ib_jr - ra_ib_j) = 0$. Then $a_ib_jr = ra_ib_j$ for all $r \in R$. Hence $a_ib_j$ is central for all $i$ and $j$. This completes the proof. ∎

Let $S$ denote a multiplicatively closed subset of a ring $R$ consisting of central regular elements. Let $S^{-1}R$ be the localization of $R$ at $S$. Then we have:

**Proposition 2.2.** *A ring $R$ is central Armendariz if and only if $S^{-1}R$ is central Armendariz.*

*Proof.* Suppose that $R$ is a central Armendariz ring. Let $f(x) = \sum_{i=0}^{s}(a_i/s_i)x^i$, $g(x) = \sum_{j=0}^{t}(b_j/t_j)x^j \in (S^{-1}R)[x]$ and $f(x)g(x) = 0$. Then we may find $u$, $v$, $c_i$ and $d_j$ in $S$ such that $uf(x) = \sum_{i=0}^{s} a_ic_ix^i \in R[x]$, $vg(x) = \sum_{i=0}^{t} b_jd_jx^j \in R[x]$ and $(uf(x))(vg(x)) = 0$. By supposition $(a_ic_i)(b_jd_j)$ are central in $R$ for all $i$ and $j$. Since $c_i$ and $d_j$ are regular elements of $R$, $a_ib_j$ are central in $R$. It follows that $(a_i/s_i)(b_j/t_j)$ are central for all $i$ and $j$.

Conversely, assume that $S^{-1}R$ is a central Armendariz ring. Since subrings of central Armendariz rings are central Armendariz and $R$ is a subring of $S^{-1}R$, $R$ is central Armendariz. ∎

**Corollary 2.2.** *For a ring $R$, the following are equivalent:*

(1) *$R$ is central Armendariz.*
(2) *$R[x]$ is central Armendariz.*
(3) *$R[x, x^{-1}]$ is central Armendariz.*

*Proof.* Let $S = \{1, x, x^2, x^3, x^4, \ldots\}$. Then $S$ is a multiplicatively closed subset of $R[x]$ consisting of central regular elements. Then the proof follows from Proposition 2.2. ∎

We end this paper with some observations concerning Baer, p.q-Baer and p.p.-rings. We show that if a ring is central Armendariz, then there is a strong connection between Baer, p.q-Baer, p.p.-rings with their polynomial rings and their Laurent polynomial rings.

**Corollary 2.3.** *Let $R$ be a central Armendariz ring. Then we have:*

(1) *$R$ is a right p.p.-ring if and only if $R[x]$ is a right p.p.-ring.*
(2) *$R$ is a Baer ring if and only if $R[x]$ is a Baer ring.*
(3) *$R$ is a right p.q.-Baer ring if and only if $R[x]$ is a right p.q.-Baer ring.*
(4) *$R$ is a Baer ring if and only if $R[[x]]$ is a Baer ring.*
(5) *$R$ is a Baer ring if and only if $R[x, x^{-1}]$ is a Baer ring.*
(6) *$R$ is a right p.p.-ring if and only if $R[x, x^{-1}]$ is a right p.p.-ring.*
(7) *$R$ is a Baer ring if and only if $R[[x, x^{-1}]]$ is a Baer ring.*

*Proof.* If the ring $R$ is central Armendariz, by Proposition 2.1, $R$ is abelian. The rest follows from [1]. ∎

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