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TIMELIKE LOXODROMES ON LORENTZIAN HELICOIDAL SURFACES IN MINKOWSKI n–SPACE

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Abstract. In this paper, we examine timelike loxodromes on three kinds of Lorentzian helicoidal surfaces in Minkowski n–space. First, we obtain the first order ordinary differential equations which determine timelike loxodromes on the Lorentzian helicoidal surfaces in $\mathbb{E}^n_1$ according to the causal characters of their meridian curves. Then, by finding general solutions, we get the explicit parametrizations of such timelike loxodromes. In particular, we investigate the timelike loxodromes on the three kinds of Lorentzian right helicoidal surfaces in $\mathbb{E}^n_1$. Finally, we give an example to visualize the results.

1. Introduction

Loxodromes, which are also known as rhumb lines, are curves that make constant angles with the meridians on the Earth's surface. Geodesics which minimize the distance between two points on Earth's surface, are different from than loxodromes on Earth's surface, [26]. Only the equator and the meridians are both constant course angle and length minimizing. Since loxodromes give an efficient routing from one position to another by means of a constant course angle, they are still primarily used in navigation. For details, we refer to [1,2,25,27]. Since the Earth's surface can be thought as a Riemannian sphere, the notion of loxodromes can be

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broaden to an arbitrary surface of revolution, where meridians are copies of the profile curve.

In early of 20th century, C. A. Noble [21] studied the loxodrome on the surface of revolution in $\mathbb{E}^3$ and he also showed that the loxodrome on sphereoid projects stereographically into the same spiral as the loxodrome on the sphere which is tangent to the sphereoid along equator. Then, S. Kos et al. [19] and M. Petrović [23] got the differential equations related to the loxodromes on a sphere and a sphereoid and determined the length of such loxodromes, respectively.

Later, the topic of loxodromes has been studied on the rotational surfaces in Minkowski space which is important in general relativity. In 3–dimensional Minkowski space, there are three types of rotational surfaces with respect to the casual characters of rotation axes and the concept of angle to define loxodromes is not similar to Riemannian case. Therefore, the results in the Minkowski space are richer than the Euclidean space. The authors determined the parametrizations of spacelike and timelike loxodromes on rotational surfaces in $\mathbb{E}^3_1$ which have either spacelike meridians or timelike meridians in [3] and [4], respectively. For 4–dimensional Minkowski space, there are three types of rotation with 2–dimensional axes such as elliptic, hyperbolic and parabolic rotation leaving a Riemannian plane, a Lorentzian plane or a degenerate plane pointwise fixed, respectively. Then, M. Babaarslan and M. Gümüş found the explicit parametrizations of loxodromes on such rotational surfaces of $\mathbb{E}^4_1$ in [10].

Helicoidal surfaces are the natural generalizations of rotational surfaces and they play important roles in nature, science and engineering, see [17, 18, 22]. Thus, this generalization leads the studies to the loxodromes on helicoidal surfaces in [5–9]. Recently, M. Babaarslan and N. Sönmez constructed the three kinds of helicoidal surfaces in $\mathbb{E}^4_1$ by using rotation with 2–dimensional axes and translation in $\mathbb{E}^4_1$ and they also obtained the general form of spacelike and timelike loxodromes on such helicoidal surfaces in [11].

With the motivation from geometry, M. Babaarslan, B. B. Demirci, and R. Genç extended the notion of the helicoidal surfaces in $\mathbb{E}^4_1$ to higher dimensional Minkowski space and they made characterization of spacelike loxodromes on these helicoidal surfaces of $\mathbb{E}^n_1$ in [12]. In this context, this paper is a sequel of the article given by [12].

In this paper, we study timelike loxodromes on three types of Lorentzian helicoidal surfaces in Minkowski n–space $\mathbb{E}^n_1$. We find the equations of timelike loxodromes on such helicodial surfaces which have either spacelike meridians or timelike meridians and then we get the explicit parametrizations of these loxodromes by finding the general solution of the equations. As particular cases, we consider timelike loxodromes on each Lorentzian right helicoidal surfaces in $\mathbb{E}^n_1$. Finally, we give an illustrative example.

2. Preliminaries

Let $\mathbb{E}^n_s$ denote the pseudo–Euclidean space of dimension n and index s, i.e., $\mathbb{R}^n = \{(x_1, x_2, ..., x_n) \mid x_1, x_2, ..., x_n \in \mathbb{R}\}$ equipped with the metric

$$ds^2 = \sum_{i=1}^{n-s} dx_i^2 - \sum_{j=n-s+1}^{n} dx_j^2. \tag{1}$$

For $s = 1$, $\mathbb{E}^n_1$ is known as the Minkowski space which is inspired by general relativity.

A vector v in $\mathbb{E}^n_1$ is called spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$, and lightlike (or null) if $\langle v, v\rangle = 0$ and $v \neq 0$. The length of a vector v in $\mathbb{E}^n_1$ is given by $||v|| = \sqrt{|\langle v, v\rangle|}$ and v is said to be an unit vector if $||v|| = 1$.

Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^n_1$ be a smooth regular curve in $\mathbb{E}^n_1$, where I is an open interval. Then, the causal character of α is spacelike, timelike or lightlike if $\dot{\alpha}$ is spacelike, timelike or lightlike, respectively, where $\dot{\alpha} = d\alpha/dt$.

Let M be a pseudo–Riemannian surface in $\mathbb{E}^n_1$ given by a local parametrization $\mathbf{x}(u, v)$. Then, the coefficients of the first fundamental form of M are

$$E = \langle \mathbf{x}_u, \mathbf{x}_u\rangle, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v\rangle, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v\rangle, \tag{2}$$

where $\mathbf{x}_u$ and $\mathbf{x}_v$ denote the partial derivatives of $\mathbf{x}$ with respect u and v, respectively. Thus, the induced metric g of M in $\mathbb{E}^n_1$ is given by

$$g = Edu^2 + 2Fdudv + Gdv^2. \tag{3}$$

Also, a pseudo–Riemannian surface M in $\mathbb{E}^n_1$ is called a spacelike surface or a timelike surface if and only if $EG - F^2 > 0$ or $EG - F^2 < 0$, respectively. For the case $EG - F^2 = 0$, a pseudo–Riemannian surface M is called a lightlike surface. Throughout this work, we will assume that the surface is nondegenerate.

The length of the curve α on the pseudo–Riemannian surface M between two points u_0 and u_1 in $\mathbb{E}^n_1$ is given by

$$L = \int_{u_0}^{u_1} \sqrt{\left| E + 2F\frac{dv}{du} + G\left(\frac{dv}{du}\right)^2 \right|} du. \tag{4}$$

For later use, we give the following definition of Lorentzian angle in $\mathbb{E}^n_1$ by using [24].

Definition 1. *Let x and y be vectors in $\mathbb{E}^n_1$. Then, we have the following statements:*

i. *for a spacelike vector x and a timelike vector y, there is a unique nonnegative real number θ such that*

$$\langle x, y\rangle = \pm||x||||y|| \sinh\theta. \tag{5}$$

The number θ is called Lorentzian timelike angle between x and y.

ii. *for timelike vectors x and y, there is a unique nonnegative real number θ such that*

$$\langle x, y\rangle = ||x||||y|| \cosh\theta. \tag{6}$$

The number θ is called Lorentzian timelike angle between x and y. Note that $\theta = 0$ if and only if x and y are positive scalar multiples of each other.

By using [12], the definition of the helicoidal surfaces in $\mathbb{E}^n_1$ can be given as follows.

Let $\beta : I \subset \mathbb{R} \longrightarrow \Pi \subset \mathbb{E}^n_1$ be a smooth curve in a hyperplane $\Pi \subset \mathbb{E}^n_1$, P be a $(n-2)$-plane in the hyperplane $\Pi \subset \mathbb{E}^n_1$ and ℓ be a line parallel to P. A helicoidal surface in $\mathbb{E}^n_1$ is defined as a rotation of the curve β around P with a translation along the line ℓ. Here, the speed of translation is proportional to the speed of this rotation. Thus, there are three types of helicoidal surfaces in $\mathbb{E}^n_1$ as follows:

2.1. **Helicoidal surface of type I.** Let $\{e_1, e_2, ..., e_n\}$ be a standard orthonormal basis for $\mathbb{E}^n_1$. Then, we choose a Lorentzian $(n-2)$–subspace $\mathbf{P}_1$ generated by $\{e_3, e_4, ..., e_n\}$, Π_1 a hyperplane generated by $\{e_1, e_3, ..., e_n\}$ and a line ℓ_1 generated by e_n. Assume that $\beta_1 : I \longrightarrow \Pi_1 \subset \mathbb{E}^n_1$, $\beta_1(u) = (x_1(u), 0, x_3(u), ..., x_n(u))$, is a smooth regular curve lying in Π_1 defined on an open interval $I \subset \mathbb{R}$ and u is arc length parameter, that is, $x'^2_1(u) + x'^2_3(u) + ... - x'^2_n(u) = \varepsilon$ with $\varepsilon = \pm 1$. For $0 \le v < 2\pi$ and a positive constant c, we consider the surface M_1

$$H_1(u, v) = (x_1(u)\cos v, x_1(u)\sin v, x_3(u), ..., x_{n-1}(u), x_n(u) + cv) \tag{7}$$

which is the parametrization of the helicoidal surface obtained the rotation of the curve β_1 that leaves the Lorentzian subspace $\mathbf{P}_1$ pointwise fixed followed by the translation along ℓ_1. The surface M_1 in $\mathbb{E}^n_1$ is called a helicoidal surface of type I. Also, the surface M_1 is called a right helicoidal surface of type I in $\mathbb{E}^n_1$ if x_n is a constant function.

2.2. **Helicoidal surface of type II.** Let $\{e_1, e_2, ..., e_n\}$ be a standard orthonormal basis for $\mathbb{E}^n_1$. Then, we choose a Riemannian $(n-2)$–subspace $\mathbf{P}_2$ generated by $\{e_1, e_2, ..., e_{n-2}\}$, Π_2 a hyperplane generated by $\{e_1, ..., e_{n-2}, e_n\}$ and a line ℓ_2 generated by e_1. Assume that $\beta_2 : I \longrightarrow \Pi_2 \subset \mathbb{E}^n_1, \beta_2(u) = (x_1(u), ..., x_{n-2}(u), 0, x_n(u))$, is a smooth regular curve lying in Π_2 defined on an open interval $I \subset \mathbb{R}$ and u is an arc length parameter, that is, $x'^2_1(u) + x'^2_2(u) + ... - x'^2_n(u) = \varepsilon$ for $\varepsilon = \pm 1$. For $v \in \mathbb{R}$ and a positive constant c, we consider the surface M_2

$$H_2(u, v) = (x_1(u) + cv, x_2(u), ..., x_{n-2}(u), x_n(u)\sinh v, x_n(u)\cosh v) \tag{8}$$

which is the parametrization of the helicoidal surface obtained the rotation of the curve β_2 which leaves Riemannian subspace $\mathbf{P}_2$ pointwise fixed followed by the translation along ℓ_2. The surface M_2 in $\mathbb{E}^n_1$ is called a helicoidal surface of type II. Also, the surface M_2 is called a right helicoidal surface of type II in $\mathbb{E}^n_1$ if x_1 is a constant function.

2.3. **Helicoidal surface of type III.** Let define a pseudo–orthonormal basis $\{e_1, e_2, ..., \xi_{n-1}, \xi_n\}$ for $\mathbb{E}_1^n$ using a standard orthonormal basis $\{e_1, e_2, ..., e_{n-1}, e_n\}$ for $\mathbb{E}_1^n$ such that

$$\xi_{n-1} = \frac{1}{\sqrt{2}}(e_n - e_{n-1}) \text{ and } \xi_n = \frac{1}{\sqrt{2}}(e_n + e_{n-1}), \tag{9}$$

where $\langle \xi_{n-1}, \xi_{n-1} \rangle = \langle \xi_n, \xi_n \rangle = 0$ and $\langle \xi_{n-1}, \xi_n \rangle = -1$. Then, we choose a degenerate $(n-2)$–subspace $\mathbf{P}_3$ generated by $\{e_1, e_3, ..., \xi_{n-1}\}$, Π_3 a hyperplane generated by $\{e_1, e_3, ..., e_{n-2}, \xi_{n-1}, \xi_n\}$ and a line ℓ_3 generated by ξ_{n-1}. Assume that $\beta_3 : I \longrightarrow \Pi_3 \subset \mathbb{E}_1^n$, $\beta_3(u) = x_1(u)e_1 + x_3(u)e_3 + ... + x_{n-1}(u)\xi_{n-1} + x_n(u)\xi_n$, is a smooth curve lying in Π_3 defined on an open interval $I \subset \mathbb{R}$ and u is an arc length parameter, that is, $x_1'^2(u) + x_3'^2(u) + ... - 2x_{n-1}'(u)x_n'(u) = \varepsilon$ for $\varepsilon = \pm 1$. Then, we consider the surface M_3

$$\begin{aligned} H_3(u,v) =& x_1(u)e_1 + \sqrt{2}vx_n(u)e_2 + x_3(u)e_3 + ... + x_{n-2}(u)e_{n-2} \\ &+ (x_{n-1}(u) + v^2x_n(u) + cv)\xi_{n-1} + x_n(u)\xi_n \end{aligned} \tag{10}$$

which is the parametrization of the helicoidal surface obtained a rotation of the curve β_3 which leaves the degenerate subspace $\mathbf{P}_3$ pointwise fixed followed by the translation along ℓ_3. The surface M_3 in $\mathbb{E}_1^n$ is called the helicoidal surface of type III. If x_n is a constant function, then the helicoidal surface M_3 is called a right helicoidal surface of type III in $\mathbb{E}_1^n$.

Remark 1. *It can be easily seen that the helicoidal surfaces* M_1–M_3 *in* $\mathbb{E}_1^n$ *defined by* (7), (8) *and* (10) *reduce to the rotational surfaces in* $\mathbb{E}_1^n$ *for* $c = 0$.

3. Timelike Loxodrome on Timelike Helicoidal Surface of Type I in $\mathbb{E}_1^n$

In this section, we determine the parametrization of timelike loxodrome on the timelike helicoidal surface of type I in $\mathbb{E}_1^n$ defined by (7).
Consider the timelike helicoidal surface of type I, M_1, in $\mathbb{E}_1^n$ given by (7). From a simple calculation, the induced metric g_1 on M_1 is defined by

$$g_1 = \varepsilon du^2 - 2cx_n'(u)dudv + (x_1^2(u) - c^2)dv^2. \tag{11}$$

Since M_1 is a timelike surface in $\mathbb{E}_1^n$, we have $\varepsilon x_1^2(u) - c^2(\varepsilon + x_n'^2(u)) < 0$. Assume that $\alpha_1(t) = H_1(u(t), v(t))$ is a timelike loxodrome on M_1 in $\mathbb{E}_1^n$, that is, $\alpha_1(t)$ intersects the meridian $m_1(u) = H_1(u, v_0)$ for a constant v_0 with a constant angle ϕ_0 at the point $p \in M_1$. Then, we have

$$\langle \dot{\alpha_1}(t), (m_1)_u \rangle = \varepsilon \frac{du}{dt} - cx_n'(u)\frac{dv}{dt}, \tag{12}$$

$$\varepsilon \left(\frac{du}{dt}\right)^2 - 2cx_n'(u)\frac{du}{dt}\frac{dv}{dt} + (x_1^2(u) - c^2)\left(\frac{dv}{dt}\right)^2 < 0. \tag{13}$$

In this context, there are two following cases occur with respect to the causal character of the meridian curve $m_1(u)$.

Case i. M_1 has a spacelike meridian curve $m_1(u)$, that is, $\varepsilon = 1$. Using the equations (12) and (13) in (5), we get

$$\sinh\phi_0 = \pm\frac{\frac{du}{dt} - cx_n'(u)\frac{dv}{dt}}{\sqrt{-\left(\frac{du}{dt}\right)^2 + 2cx_n'(u)\frac{du}{dt}\frac{dv}{dt} - (x_1^2(u) - c^2)\left(\frac{dv}{dt}\right)^2}}. \tag{14}$$

Case ii. M_1 has a timelike meridian curve $m_1(u)$, that is, $\varepsilon = -1$. Using the equations (12) and (13) in (6), we obtain

$$\cosh\phi_0 = -\frac{\frac{du}{dt} + cx_n'(u)\frac{dv}{dt}}{\sqrt{\left(\frac{du}{dt}\right)^2 + 2cx_n'(u)\frac{du}{dt}\frac{dv}{dt} - (x_1^2(u) - c^2)\left(\frac{dv}{dt}\right)^2}}. \tag{15}$$

After a simple calculation in equations (14) and (15), we get the following lemma.

Lemma 1. *Let M_1 be a timelike helicoidal surface of type I in $\mathbb{E}_1^n$ defined by (7). Then, $\alpha_1(t) = H_1(u(t), v(t))$ is a timelike loxodrome with $\dot{u} \neq 0$ if and only if one of the following differential equations is satisfied:*

(i.) *for having a spacelike meridian,*

$$(\sinh^2\phi_0(x_1^2(u) - c^2) + c^2x_n'^2(u))\dot{v}^2 - 2c\cosh^2\phi_0 x_n'(u)\dot{u}\dot{v} + \cosh^2\phi_0\dot{u}^2 = 0, \tag{16}$$

(ii.) *for having a timelike meridian,*

$$(\cosh^2\phi_0(x_1^2(u) - c^2) + c^2x_n'^2(u))\dot{v}^2 - 2c\sinh^2\phi_0 x_n'(u)\dot{u}\dot{v} - \sinh^2\phi_0\dot{u}^2 = 0, \tag{17}$$

where ϕ_0 is a nonnegative constant.

Theorem 1. *A timelike loxodrome on a timelike helicoidal surface of type I in $\mathbb{E}_1^n$ defined by (7) is parametrized by $\alpha_1(u) = H_1(u, v(u))$, where $v(u)$ is given by one of the following functions:*

(i.) $v(u) = \pm\dfrac{1}{2\sinh\phi_0}\displaystyle\int_{u_0}^{u}\frac{d\xi}{\sqrt{c^2 - x_1^2(\xi)}},$

(ii.) $v(u) = \pm\dfrac{1}{2\cosh\phi_0}\displaystyle\int_{u_0}^{u}\frac{d\xi}{\sqrt{c^2 - x_1^2(\xi)}},$

(iii.) *for* $\sinh^2\phi_0(x_1^2(\xi) - c^2) + c^2x_n'^2(\xi) \neq 0,$

$$v(u) = \int_{u_0}^{u}\frac{2c\cosh^2\phi_0 x_n'(\xi) \pm \sqrt{\sinh^2(2\phi_0)(c^2(x_n'^2(\xi) + 1) - x_1^2(\xi))}}{2\sinh^2\phi_0(x_1^2(\xi) - c^2) + 2c^2x_n'^2(\xi)}d\xi,$$

(iv.) *for* $\cosh^2\phi_0(x_1^2(\xi) - c^2) + c^2x_n'^2(\xi) \neq 0,$

$$v(u) = \int_{u_0}^{u}\frac{2c\sinh^2\phi_0 x_n'(\xi) \pm \sqrt{\sinh^2(2\phi_0)(c^2(x_n'^2(\xi) - 1) + x_1^2(\xi))}}{2\cosh^2\phi_0(x_1^2(\xi) - c^2) + 2c^2x_n'^2(\xi)}d\xi,$$

where ϕ_0 is a nonnegative constant and $c > 0$ is a constant.

Proof. Assume that M_1 is a timelike helicoidal surface in $\mathbb{E}_1^n$ defined by (7) and $\alpha_1(t) = H_1(u(t), v(t))$ is a timelike loxodrome on M_1 in $\mathbb{E}_1^n$. From Lemma 1, we have the equations (16) and (17).

For a spacelike meridian, the equation (16) implies

$$(\sinh^2\phi_0(x_1^2(u)-c^2)+c^2x_n'^2(u))\left(\frac{dv}{du}\right)^2-2c\cosh^2\phi_0x_n'(u)\frac{dv}{du}+\cosh^2\phi_0=0. \quad (18)$$

If $\sinh^2\phi_0(x_1^2(u)-c^2)+c^2x_n'^2(u)=0$, then the equation (18) becomes

$$2c\cosh^2\phi_0x_n'(u)\frac{dv}{du}-\cosh^2\phi_0=0 \quad (19)$$

whose the solution is $v(u)=\frac{1}{2c}\int_{u_0}^{u}\frac{d\xi}{x_n'(\xi)}$. On the other side, $\sinh^2\phi_0(x_1^2(u)-c^2)+c^2x_n'^2(u)=0$ implies $x_n'(u)=\pm\frac{\sinh\phi_0}{c}\sqrt{c^2-x_1^2(u)}$ for $\phi_0\neq0$. Thus, we get the desired equation in (i). Also, we note that $c^2-x_1^2(u)>0$ due the fact that M_1 is a timelike surface in $\mathbb{E}_1^n$.

If $\sinh^2\phi_0(x_1^2(u)-c^2)+c^2x_n'^2(u)\neq0$, it can be easily obtained that the solution $v(u)$ of the differential equation (18) is given by the integral in (iii).

Similarly, for a timelike meridian, the equation (17) implies

$$(\cosh^2\phi_0(x_1^2(u)-c^2)+c^2x_n'^2(u))\left(\frac{dv}{du}\right)^2-2c\sinh^2\phi_0x_n'(u)\frac{dv}{du}-\sinh^2\phi_0=0. \quad (20)$$

If $\cosh^2\phi_0(x_1^2(u)-c^2)+c^2x_n'^2(u)=0$, the equation (20) reduces to the following equation

$$2c\sinh^2\phi_0x_n'(u)\frac{dv}{du}+\sinh^2\phi_0=0 \quad (21)$$

whose the solution is $v(u)=-\frac{1}{2c}\int_{u_0}^{u}\frac{d\xi}{x_n'(\xi)}$ for a nonzero constant ϕ_0. Since $x_n'(u)=\pm\frac{\cosh\phi_0}{c}\sqrt{c^2-x_1^2(u)}$, we get the desired equation in (ii). Also, we note that $c^2-x_1^2(u)>0$ due the fact that M_1 is a timelike surface in $\mathbb{E}_1^n$. If $\cosh^2\phi_0(x_1^2(u)-c^2)+c^2x_n'^2(u)\neq0$, the solution $v(u)$ of the differential equation (20) is given by the integral in (iv). Thus, we get the parametrization of the loxodrome with respect to u parameter such that $\alpha_1(u)=H_1(u,v(u))$, where $v(u)$ is defined by one of the integrals in (i)-(iv). □

Now, we consider a timelike right helicoidal surface of type I in $\mathbb{E}_1^n$, denoted by M_1^R, that is,

$$H_1^R(u,v)=(x_1(u)\cos v,x_1(u)\sin v,x_3(u),\ldots,x_{n-1}(u),x_{n_0}+cv), \quad (22)$$

where $c\neq0$ and x_{n_0} are constants. Then, from the equation in (iii) of Theorem 1, we give the following corollary.

Corollary 1. *A timelike loxodrome on a timelike right helicoidal surface of type I in $\mathbb{E}_1^n$ defined by (22) is parametrized by $\alpha_1^R(u)=H_1^R(u,v(u))$ where $v(u)$ is given by*

$$v(u)=\pm\coth\phi_0\int_{u_0}^{u}\frac{d\xi}{\sqrt{c^2-x_1^2(\xi)}} \quad (23)$$

for constant $\phi_0>0$.

Using the equation (4) and Corollary 1, we give the following statement:

Corollary 2. *The length of a timelike loxodrome on a timelike right helicoidal surface of type I in* $\mathbb{E}_1^n$ *defined by* (22) *between two points* u_0 *and* u_1 *is given by*

$$L = \left| \frac{u_1 - u_0}{\sinh \phi_0} \right|,$$

for constant $\phi_0 > 0$.

4. Timelike Loxodrome on Timelike Helicoidal Surface of Type II in $\mathbb{E}_1^n$

In this section, we determine the parametrization of timelike loxodrome on the timelike helicoidal surface of type II in $\mathbb{E}_1^n$ defined by (8).

Consider the timelike helicoidal surface of type II, M_2, in $\mathbb{E}_1^n$ given by (8). From a simple calculation, the induced metric g_2 on M_2 is defined by

$$g_2 = \varepsilon du^2 + 2cx_1'(u)dudv + (c^2 + x_n^2(u))dv^2. \tag{24}$$

Since M_2 is a timelike surface in $\mathbb{E}_1^n$, we have $c^2(\varepsilon - x_1'^2(u)) + \varepsilon x_n^2(u) < 0$. Assume that $\alpha_2(t) = H_2(u(t), v(t))$ is a timelike loxodrome on M_2 in $\mathbb{E}_1^n$, that is, $\alpha_2(t)$ intersects the meridian $m_2(u) = H_2(u, v_0)$ for a constant v_0 with a constant angle ϕ_0 at the point $p \in M_2$. Then, we have

$$\langle \dot{\alpha}_2(t), (m_2)_u \rangle = \varepsilon \frac{du}{dt} + cx_1'(u)\frac{dv}{dt}, \tag{25}$$

$$\varepsilon \left(\frac{du}{dt}\right)^2 + 2cx_1'(u)\frac{du}{dt}\frac{dv}{dt} + (c^2 + x_n^2(u)) \left(\frac{dv}{dt}\right)^2 < 0. \tag{26}$$

In this context, there are two following cases occur with respect to the causal character of the meridian curve $m_2(u)$.
Case i. M_2 has a spacelike meridian curve $m_2(u)$, that is, $\varepsilon = 1$. Using the equations (25) and (26) in (5), we get

$$\sinh \phi_0 = \pm \frac{\frac{du}{dt} + cx_1'(u)\frac{dv}{dt}}{\sqrt{-\left(\frac{du}{dt}\right)^2 - 2cx_1'(u)\frac{du}{dt}\frac{dv}{dt} - (c^2 + x_n^2(u))\left(\frac{dv}{dt}\right)^2}}. \tag{27}$$

Case ii. M_2 has a timelike meridian curve $m_2(u)$, that is, $\varepsilon = -1$. Using the equations (25) and (26) in (6), we obtain

$$\cosh \phi_0 = \frac{-\frac{du}{dt} + cx_1'(u)\frac{dv}{dt}}{\sqrt{\left(\frac{du}{dt}\right)^2 - 2cx_1'(u)\frac{du}{dt}\frac{dv}{dt} - (c^2 + x_n^2(u))\left(\frac{dv}{dt}\right)^2}}. \tag{28}$$

After a simple calculation in equations (27) and (28), we get the following lemma.

Lemma 2. *Let* M_2 *be a timelike helicoidal surface of type II in* $\mathbb{E}_1^n$ *defined by* (8). *Then,* $\alpha_2(t) = H_2(u(t), v(t))$ *is a timelike loxodrome with* $\dot{u} \neq 0$ *if and only if one of the following differential equations is satisfied:*

(i.) *for having a spacelike meridian,*

$$(\sinh^2\phi_0(x_n^2(u)+c^2)+c^2x_1'^2(u))\dot{v}^2+2c\cosh^2\phi_0 x_1'(u)\dot{u}\dot{v}+\cosh^2\phi_0\dot{u}^2=0, \tag{29}$$

(ii.) *for having a timelike meridian,*

$$(\cosh^2\phi_0(x_n^2(u)+c^2)+c^2x_1'^2(u))\dot{v}^2+2c\sinh^2\phi_0 x_1'(u)\dot{u}\dot{v}-\sinh^2\phi_0\dot{u}^2=0, \tag{30}$$

where ϕ_0 is a nonnegative constant.

Theorem 2. *A timelike loxodrome on a timelike helicoidal surface of type II in $\mathbb{E}_1^n$ defined by* (8) *is parametrized by $\alpha_2(u)=H_2(u,v(u))$, where $v(u)$ is given by one of the following functions:*

(i.) $$v(u)=\int_{u_0}^{u}\frac{-2c\cosh^2\phi_0 x_1'(\xi)\pm\sqrt{\sinh^2(2\phi_0)(c^2(x_1'^2(\xi)-1)-x_n^2(\xi))}}{2\sinh^2\phi_0(x_n^2(\xi)+c^2)+2c^2x_1'^2(\xi)}d\xi,$$

(ii.) $$v(u)=\int_{u_0}^{u}\frac{-2c\sinh^2\phi_0 x_1'(\xi)\pm\sqrt{\sinh^2(2\phi_0)(x_n^2(\xi)+c^2(x_1'^2(\xi)+1))}}{2\cosh^2\phi_0(x_n^2(\xi)+c^2)+2c^2x_1'^2(\xi)}d\xi,$$

where ϕ_0 is a nonnegative constant.

Proof. Assume that M_2 is a timelike helicoidal surface in $\mathbb{E}_1^n$ defined by (8) and $\alpha_2(t)=H_2(u(t),v(t))$ is a timelike loxodrome on M_2 in $\mathbb{E}_1^n$. From Lemma 2, we have the equations (29) and (30).

For a spacelike meridian, the equation (29) implies

$$(\sinh^2\phi_0(x_n^2(u)+c^2)+c^2x_1'^2(u))\left(\frac{dv}{du}\right)^2+2c\cosh^2\phi_0 x_1'(u)\frac{dv}{du}+\cosh^2\phi_0=0. \tag{31}$$

Since $\sinh^2\phi_0(x_n^2(u)+c^2)+c^2x_1'^2(u)\neq 0$ for all $u\in I\subset\mathbb{R}$, it can be easily obtained that the solution $v(u)$ of the differential equation (31) is given by the integral in (i).

Similarly, for a timelike meridian, the equation (30) implies

$$(\cosh^2\phi_0(x_n^2(u)+c^2)+c^2x_1'^2(u))\left(\frac{dv}{du}\right)^2+2c\sinh^2\phi_0 x_1'(u)\frac{dv}{du}-\sinh^2\phi_0=0. \tag{32}$$

Due to $\cosh^2\phi_0(x_n^2(u)+c^2)+c^2x_1'^2(u)\neq 0$, the solution $v(u)$ of the differential equation (32) is given by the integral in (ii). Thus, we get a parametrization of the loxodrome with respect to u parameter such that $\alpha_2(u)=H_2(u,v(u))$, where $v(u)$ is defined by one of the integrals in (i) and (ii). □

Now, we consider a timelike right helicoidal surface of type II in $\mathbb{E}_1^n$ denoted by M_2^R, that is,

$$H_2^R(u,v)=(x_{1_0}+cv,x_2(u),\dots,x_{n-2}(u),x_n(u)\sinh v,x_n(u)\cosh v), \tag{33}$$

where $c\neq 0$ and x_{1_0} are constants. Since M_2^R is a timelike surface in $\mathbb{E}_1^n$, we have $\varepsilon(c^2+x_n^2(u))<0$. This inequality can be only satisfied when $\varepsilon=-1$. Thus, the meridian curve of M_2^R must be timelike. Then, from the equations in (ii) of Theorem 2, we give the following corollary.

Corollary 3. *A timelike loxodrome on a timelike right helicoidal surface of type II in $\mathbb{E}_1^n$ defined by (33) is parametrized by $\alpha_2^R(u) = H_2^R(u, v(u))$, where $v(u)$ is given by*

$$v(u) = \pm \tanh\phi_0 \int_{u_0}^{u} \frac{d\xi}{\sqrt{x_n^2(\xi) + c^2}} \tag{34}$$

and $c, \phi_0 > 0$ are constants.

Using the equation (4) and Corollary 3, we give the following statement:

Corollary 4. *The length of a timelike loxodrome on a timelike right helicoidal surface of type II in $\mathbb{E}_1^n$ defined by (33) between two points u_0 and u_1 is given by*

$$L = \left| \frac{u_1 - u_0}{\cosh\phi_0} \right|, \tag{35}$$

where ϕ_0 is a nonnegative constant.

5. Timelike Loxodrome on Timelike Helicoidal Surface of Type III in $\mathbb{E}_1^n$

In this section, we determine the parametrization of timelike loxodrome on the timelike helicoidal surface of type III in $\mathbb{E}_1^n$ defined by (10).

Consider the timelike helicoidal surface of type III, M_3, in $\mathbb{E}_1^n$ given by (10). The induced metric g_3 on M_3 is defined by

$$g_3 = \varepsilon du^2 - 2cx_n'(u)dudv + 2x_n^2(u)dv^2. \tag{36}$$

Since M_3 is a timelike surface in $\mathbb{E}_1^n$, we have $2\varepsilon x_n^2(u) - c^2 x_n'^2(u) < 0$. Assume that $\alpha_3(t) = H_3(u(t), v(t))$ is a timelike loxodrome on M_3 in $\mathbb{E}_1^n$, that is, $\alpha_3(t)$ intersects the meridian $m_3(u) = H_3(u, v_0)$ for a constant v_0 with a constant angle ϕ_0 at the point $p \in M_3$. Then, we have

$$\langle \dot{\alpha}_3(t), (m_3)_u \rangle = \varepsilon \frac{du}{dt} - cx_n'(u)\frac{dv}{dt}, \tag{37}$$

$$\varepsilon \left(\frac{du}{dt}\right)^2 - 2cx_n'(u)\frac{du}{dt}\frac{dv}{dt} + 2x_n^2(u)\left(\frac{dv}{dt}\right)^2 < 0. \tag{38}$$

In this context, there are two following cases occur with respect to the causal character of the meridian curve $m_3(u)$.

Case i. M_3 has a spacelike meridian curve $m_3(u)$, that is, $\varepsilon = 1$. Using the equations (37) and (38) in (5), we get

$$\sinh\phi_0 = \pm \frac{\frac{du}{dt} - cx_n'(u)\frac{dv}{dt}}{\sqrt{-\left(\frac{du}{dt}\right)^2 + 2cx_n'(u)\frac{du}{dt}\frac{dv}{dt} - 2x_n^2(u)\left(\frac{dv}{dt}\right)^2}}. \tag{39}$$

Case ii. M_3 has a timelike meridian curve $m_3(u)$, that is, $\varepsilon = -1$. Using the equations (37) and (38) in (6), we obtain

$$\cosh\phi_0 = -\frac{\frac{du}{dt} + cx_n'(u)\frac{dv}{dt}}{\sqrt{\left(\frac{du}{dt}\right)^2 + 2cx_n'(u)\frac{du}{dt}\frac{dv}{dt} - 2x_n^2(u)\left(\frac{dv}{dt}\right)^2}}. \tag{40}$$

After a simple calculation in the equations (39) and (40), we get the following lemma.

Lemma 3. *Let M_3 be a timelike helicoidal surface of type III in $\mathbb{E}_1^n$ defined by* (10). *Then, $\alpha_3(t) = H_3(u(t), v(t))$ is a timelike loxodrome with $\dot{u} \neq 0$ if and only if one of the following differential equations is satisfied:*

(i.) *for having a spacelike meridian,*

$$(2\sinh^2\phi_0 x_n^2(u) + c^2x_n'^2(u))\dot{v}^2 - 2c\cosh^2\phi_0 x_n'(u)\dot{u}\dot{v} + \cosh^2\phi_0\dot{u}^2 = 0, \tag{41}$$

(ii.) *for having a timelike meridian,*

$$(2\cosh^2\phi_0 x_n^2(u) + c^2x_n'^2(u))\dot{v}^2 - 2c\sinh^2\phi_0 x_n'(u)\dot{u}\dot{v} - \sinh^2\phi_0\dot{u}^2 = 0, \tag{42}$$

where ϕ_0 is a nonnegative constant.

Theorem 3. *A timelike loxodrome on a timelike helicoidal surface of type III in $\mathbb{E}_1^n$ defined by* (10) *is parametrized by $\alpha_3(u) = H_3(u, v(u))$, where $v(u)$ is given by one of the following functions:*

(i.) $$v(u) = \int_{u_0}^{u} \frac{2c\cosh^2\phi_0 x_n'(\xi) \pm \sqrt{\sinh^2(2\phi_0)(c^2x_n'^2(\xi) - 2x_n^2(\xi))}}{4\sinh^2\phi_0 x_n^2(\xi) + 2c^2x_n'^2(\xi)}d\xi,$$

(ii.) $$v(u) = \int_{u_0}^{u} \frac{2c\sinh^2\phi_0 x_n'(\xi) \pm \sqrt{\sinh^2(2\phi_0)(2x_n^2(\xi) + c^2x_n'^2(\xi))}}{4\cosh^2\phi_0 x_n^2(\xi) + 2c^2x_n'^2(\xi)}d\xi,$$

where ϕ_0 is a nonnegative constant.

Proof. Assume that M_3 is a timelike helicoidal surface in $\mathbb{E}_1^n$ defined by (10) and $\alpha_3(t) = H_3(u(t), v(t))$ is a timelike loxodrome on M_3 in $\mathbb{E}_1^n$. From Lemma 3, we have the equations (41) and (42).

For a spacelike meridian, the equation (41) implies

$$(2\sinh^2\phi_0 x_n^2(u) + c^2x_n'^2(u))\left(\frac{dv}{du}\right)^2 - 2c\cosh^2\phi_0 x_n'(u)\frac{dv}{du} + \cosh^2\phi_0 = 0. \tag{43}$$

Since $2\sinh^2\phi_0 x_n^2(u) + c^2x_n'^2(u) \neq 0$ for all $u \in I \subset \mathbb{R}$, it can be easily obtained that the solution $v(u)$ of the differential equation (43) is given by the integral in (i).

Similarly, for a timelike meridian, the equation (42) implies

$$(2\cosh^2\phi_0 x_n^2(u) + c^2x_n'^2(u))\left(\frac{dv}{du}\right)^2 - 2c\sinh^2\phi_0 x_n'(u)\frac{dv}{du} - \sinh^2\phi_0 = 0. \tag{44}$$

Due to $2\cosh^2\phi_0 x_n^2(u)+c^2x_n'^2(u)\neq 0$, the solution $v(u)$ of the differential equation (44) is given by the integral in (ii). Thus, we get the parametrization of the loxodrome with respect to u parameter such that $\alpha_3(u)=H_3(u,v(u))$, where $v(u)$ is defined by one of the integrals in (i) and (ii). □

Note that the timelike right helicoidal surface of type III with the timelike meridian does not exist.

6. Visualization

In this section, we give an example to visualize our main results.

Example 1. *We consider the following spacelike profile curve:*

$$\beta_1(u)=(x_1(u),0,x_3(u),...,x_n(u)).$$

Then, we have the following parametrization of timelike helicoidal surface M_1*:*

$$H_1(u,v)=(x_1(u)\cos v, x_1(u)\sin v, x_3(u),...,x_{n-1}(u),x_n(u)+cv).$$

By using (i) of Theorem 1, we have $v(u)=\pm\frac{1}{2\sinh\phi_0}\int_{u_0}^{u}\frac{d\xi}{\sqrt{c^2-x_1^2(\xi)}}$*. If we choose* $x_1(\xi)=ck\sin\xi$ *for* $0<k<1$*, then* $v(u)=\pm\frac{1}{2c\sinh\phi_0}\int_{u_0}^{u}\frac{d\xi}{\sqrt{1-k^2\sin^2\xi}}=\pm\frac{1}{2c\sinh\phi_0}F(u,k)$*, where* $F(u,k)$ *is an elliptic integral of first kind (see [13]). Then, the parametrization of timelike loxodrome on timelike helicoidal surface* M_1 *in Minkowski* n*–space is given by*

$$\alpha_1(u)=(x_1(u)\cos v(u), x_1(u)\sin v(u), x_3(u),...,x_{n-1}(u),x_n(u)+cv(u)),$$

where $v(u)=\pm\frac{1}{2c\sinh\phi_0}F(u,k)$ *for* $0<k<1$*.*

7. Conclusion

Loxodromes on various surfaces and hypersurfaces in different ambient spaces have been studied and many significant results have been obtained, see [3, 14–16, 20, 21, 28]. In this paper, we investigate the timelike loxodromes on Lorentzian helicoidal surfaces in Minkowski n–space which were constructed in [12], called type I, type II and type III. For this reason, we get the first order ordinary differential equations which determine the parametrizations of the timelike loxodromes on such helicoidal surfaces. Solving these equations, we obtain the explicit parametrizations of the such loxodromes parametrized by the parameter of the profile curves of the helicoidal surfaces. It is known that a particular case of helicoidal surfaces is right helicoidal surfaces. We observe that the Lorentzian right helicoidal surfaces appear only for the Lorentzian helicoidal surfaces of type I having spacelike meridians and the Lorentzian helicoidal surfaces of type II having timelike meridians. Hence, we look the parametrizations for timelike loxodromes on which the Lorentzian right helicoidal of $\mathbb{E}_1^n$ exist. Moreover, we find the lengths of such loxodromes which just depend on the points and the angle between the loxodromes and the meridians of the surfaces. Finally, we give a theoretical example to give the concept of the loxodromes. In [11], the graphical examples of the loxodromes can be found for the 4–dimensional Minkowski space. Hence, our results in this paper and [12] can be used as finding the parametrizations of spacelike and timelike loxodromes on the nondegenerate helicoidal surfaces in the Minkowski space with the higher dimension than four.

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