Hindawi
Journal of Mathematics
Volume 2022, Article ID 1414415, 8 pages
https://doi.org/10.1155/2022/1414415

Research Article

The Best Approximation of Generalized Fuzzy Numbers Based on Scaled Metric

Tofigh Allahviranloo,[1] **Rasoul Saneifard**,[2] **Rahim Saneifard**,[3] **Farzad Kiani**,[4] **Samad Noeiaghdam**,[5,6] **and Vediyappan Govindan**[7]

[1]*Faculty of Engineering and Natural Sciences, Istinye University, Istanbul, Turkey*
[2]*Department of Engineering, Texas Southern University, Houston, Texas, USA*
[3]*Department of Mathematics, Urmia Branch, Islamic Azad University, Urmia, Iran*
[4]*Computer Engineering Department, Faculty of Engineering, Fatih Sultan Mehmet Vakif University, Istanbul, Turkey*
[5]*Industrial Mathematics Laboratory, Baikal School of BRICS, Irkutsk National Research Technical University, Irkutsk 664074, Russia*
[6]*Department of Applied Mathematics and Programming, South Ural State University, Lenin Prospect 76, Chelyabinsk 454080, Russia*
[7]*Department of Mathematics, DMI St. John the Baptist University, Mangochi, P.O. Box 406, Malawi*

Correspondence should be addressed to Vediyappan Govindan; govindoviya@gmail.com

Received 16 April 2022; Revised 21 October 2022; Accepted 4 November 2022; Published 15 November 2022

Academic Editor: Mehdi Ghatee

The ongoing study has been vehemently allocated to propound an ameliorated α-weighted generalized approximation of an arbitrary fuzzy number. This method sets out to lessen the distance between the original fuzzy set and its approximation. In an effort to elaborate the study, formulas are designed for computing the ameliorated approximation by using a multitude of examples. The numerical samples will be exemplified to illuminate the improvement of the nearest triangular approximation (Abbasbandy et al., Triangular approximation of fuzzy numbers using α-weighted valuations, *Soft Computing*, 2019). A variety of features of the ameliorated approximation are then proved.

1. Introduction

Fuzzy numbers have matured in many disciplines dealing with vagueness and imperfect information in decision-making, expert systems, and determining other related areas.

The importance of a fuzzy number has been recognized considering the uppermost representation of information is numeric. Since its inception, a large circle of researchers has focused on refinements of computing to the nearest interval of triangular, trapezoidal, or parametric approximation of a fuzzy number. New developments, to the extent that they are relevant to this field, have been served very fast. Defuzzification methods have been extensively surveyed and applied in areas of applications such as fuzzy expert systems. The noteworthy idea of defuzzification is expected to encapsulate a fuzzy number to get a typical value from a given fuzzy set. The end goal is to deliver a correspondence from the set of all fuzzy quantities into a set of crisp values. Recently, a wide range of defuzzification procedures has been proposed in the literature that should facilitate the study [1]. Detyniecki and Yager have studied the ranking fuzzy numbers using α-weighted valuations in [2]. In [3], the nearest interval approximation of a fuzzy number has been discussed by Grzegorzewski. A novel approach of defuzzification is presented by Ma et al. [4]. Yeger and Filev [5, 6] have studied the ranking fuzzy numbers using valuations and instantiation of possibility distributions.

Carlsson et al. announced an interval-valued possibilistic mean [7], and Chanas explicated an interval approximation of a fuzzy number [8]. Abbasbandy and Asady featured a procedure of the nearest trapezoidal fuzzy number to a general fuzzy number [9]. In addition, Delgado et al. [1] and

Abbasbandy et al. [10] circulated symmetrical and non-symmetrical trapezoidal representation of fuzzy numbers and triangular approximations of fuzzy numbers. The simple and more assessment method applied in these works needs to be established as a moderation of a distance between an arbitrary fuzzy number and its nearest approximation.

The previously mentioned authors examined the advanced generalized approximation of fuzzy quantities assumed by introducing a metric on the set of fuzzy numbers in a minimization process. The expected result was the computation of the core of the triangular fuzzy quantity. To conduct the study, the researcher goes through the following sections. In Section 2, we are going to expound a number of notions of fuzzy numbers and announce a weighted distance between fuzzy numbers and the refined proposed approximation method in [10]. In Section 3, the weighted triangular approximation of a fuzzy number will be investigated in terms of weighted distance to reach its general calculating formula. It is advantageous and absolutely necessary to go through a number of reasonable properties in Section 4. Examples are served in Section 5. Section 6 comprises a conclusion. Numerous references are included in the final section.

2. Preliminaries

The vital definitions of a fuzzy number [10] have been created as follows:

Definition 1. Let $\mathfrak{R}$ be the set of whole real numbers. A fuzzy number is a fuzzy set such as $A: \mathfrak{R} \longrightarrow I = [0,1]$ that satisfies

(1) A is upper semicontinuous
(2) $A(x) = 0$ outside various interval $[a, d]$
(3) There are real numbers b, c such that $a \le b \le c \le d$ then
  (a) $A(x)$ escalates on $[a, b]$
  (b) $A(x)$ descend on $[c, d]$
  (c) $A(x) = 1$, $b \le x \le c$

Definition 2. A crucial sample of a fuzzy quantity is the generalized fuzzy quantity u centered at u_C with widths $\sigma > 0$ and $\beta > 0$.

$$\begin{cases} \dfrac{1}{\sigma}\left(x - u_c + \sigma\right), & u_c \le x \le u_c + \beta, \\ \dfrac{1}{\beta}\left(u_c + \beta - x\right), & u_c - \sigma \le x \le u_c, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The parametric form of the (1) is $\underline{u}(\alpha) = u_c - \sigma + \sigma\alpha$ and $\overline{u}(\alpha) = u_c + \beta - \beta\alpha$. If the triangular fuzzy number in E is contributed by an ordered triple $u = (u_l, u_c, u_r) \in \mathfrak{R}^3$ in particular $u_l \le u_c \le u_r$, then $[u]_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)] = [u_c - (1 - \alpha)(u_c - u_l), u_c + (1 - \alpha)(u_r - u_c)]$.

2.1. Assessment Procedure. In this section, we provide the particular valuation methods. It is often too appropriate to deal with the problem of comparison of fuzzy numbers associating with a fuzzy number u some representative value, $\operatorname{Val}(u)$. There is a conspicuous case in point to compare the fuzzy subsets in terms of these single representative values.

$$\operatorname{Val}(u) = \int_0^1 \operatorname{Average}\left([u]_\alpha\right) d\alpha. \tag{2}$$

Equation (2) is elicited. A popularized classification for a set of assessment relations,

$$\operatorname{Val}(u) = \frac{\int_0^1 \operatorname{Average}\left([u]_\alpha\right) f(\alpha) d\alpha}{\int_0^1 f(\alpha) d\alpha}, \tag{3}$$

where f is a mapping from $[0,1]$ to $[0,1]$. In the increasing family, we take it to the consideration $f: \alpha \longrightarrow f(\alpha) = \alpha^q$, by $q \ge 0$. Two satisfying specific cases are about to be observed (the extremes):

(i) For $q = 0$, we observe equal to 1. Thereupon, the valuation is given by the original Equation (2).

(ii) If $q \longrightarrow \infty$, the Dirac relation is attained. As a result, the valuation will be

$$\operatorname{Val}(u) = \operatorname{Average}\left([u]_1\right), \tag{4}$$

which is categorized in place of the average of the core. Conforming to our investigation, it is highly desirable to pay extra attention to the higher α-level sets as we have the larger q. In the limiting case, when q approaches infinity, we just use u_1 (the core of u). The decreasing family which is the complementary case of the increasing family is shown as follows:

$$f: \alpha \longrightarrow f(\alpha) = (1 - \alpha)^q, \quad \text{by} q \ge 0. \tag{5}$$

Here, anew, the authors have two favorite instances (the extremity):

(i) For $q = 0$, the authors obtain f equal to 1. The estimate expressed in Equation (2) will be promised.

(ii) For $q \longrightarrow \infty$, we obtain the Dirac function. This implies that the valuation is expressed as follows:

$$\operatorname{Val}(u) = \operatorname{Average}\left([u]_0\right), \tag{6}$$

which is the average of the support. In the decreasing family, for $q \ge 0$, we are placing more emphasis on the lower α-level sets. The larger q has more emphasis given to the lower $\alpha-$ cut sets.

The best generalized fuzzy quantities $T(u)$ to u in [10] can be shown as follows:

$$\begin{cases} t_l = \dfrac{1}{6+4w}\left\{6B - 2wt_c - 9\displaystyle\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + (9+12w)\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\right\}, \\ t_c = \mathrm{Val}(u) = \mathrm{Average}([u]_1), \\ t_r = \dfrac{1}{6+4w}\left\{6B - 2wt_c - 9\displaystyle\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + (9+12w)\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\right\}. \end{cases} \tag{7}$$

Such that $B = \mathrm{Val}(u)$.

3. The Best Generalized Fuzzy Number

We shall now propose the triangular approximation of a fuzzy number by the minimization technique, and we will use a metric to calculate the core of the generalized fuzzy quantity $T(u)$. Taking u as a general fuzzy number, this paper is trying to probe the nearest generalized fuzzy quantity:

$$T(u) = (t_l(u), t_c(u), t_r(u)) = (t_l, t_c, t_r). \tag{8}$$

Initially, we compute the core of $T(u)$. This process can be extended with respect to the following minimization technique:

$$\mathrm{Minimize} S = [\mathrm{Val}(u) - \mathrm{Val}(T(u))]^2. \tag{9}$$

With (3) and parametric form of $T(u)$ as $(\underline{T}(u)(\alpha), \overline{T}(u)(\alpha))$, subsequent form is provided as follows:

$$\begin{aligned} \mathrm{Val}(T(u)) &= \frac{\int_0^1 (\underline{T}(u)(\alpha) + \overline{T}(u)(\alpha)/2)\alpha^q \mathrm{d}\alpha}{\int_0^1 \alpha^q \mathrm{d}\alpha} \\ &= \frac{1}{2}\frac{\int_0^1 [2t_c + (1-\alpha)(t_r - 2t_c + t_l)]\alpha^q \mathrm{d}\alpha}{\int_0^1 \alpha^q \mathrm{d}\alpha} \\ &= \frac{q+1}{2}\left[\frac{2t_c}{q+1} + (t_r - 2t_c + t_l)\left(\frac{1}{q+1} - \frac{1}{q+2}\right)\right] \\ &= t_c + \frac{(t_r - 2t_c + t_l)}{2(q+2)}. \end{aligned} \tag{10}$$

To accomplish the core of $T(u)$, the partial derivative will be shown as follows:

$$\frac{\partial s}{\partial t_c} = 2\left(\mathrm{Val}(u) - t_c - \frac{(t_r - 2t_c + t_l)}{2(q+2)}\right)\left(-1 + \frac{1}{q+2}\right). \tag{11}$$

And we solve $\partial s/\partial t_c = 0$ which results in

$$t_c = \mathrm{Val}(u)\frac{q+2}{q+1} - \frac{(t_r - t_l)}{2(q+1)}. \tag{12}$$

Looking at the 1-cut set, and extra q, the higher level sets would be produced when $f(\alpha) = \alpha^q$, the core of $T(u)$ to be clarified. This is the case when $q \longrightarrow \infty$ one obtains

$$t_c = \mathrm{Val}(u) = \mathrm{Average}([u]_1). \tag{13}$$

To calculate the core of the generalized fuzzy quantity $T(u)$, the subsequent mathematical programming problem is constructed:

$$\begin{aligned} \mathrm{Min} P &= D_1^2(u, T(u)) + wD_2^2(u, T(u)), \\ D_1(u, T(u)) &= |\mathrm{Val}(u) - \mathrm{Val}(T(u))|. \end{aligned} \tag{14}$$

Giving consideration to

$$D_2(u, T(u)) = \left(\int_0^1 f(\alpha)[\underline{u}(\alpha) - \underline{T}(u)(\alpha)]^2 \mathrm{d}\alpha + \int_0^1 f(\alpha)[\overline{u}(\alpha) - \overline{T}(u)(\alpha)]^2 \mathrm{d}\alpha\right)^{1/2}. \tag{15}$$

And $w \geq 0$ is an extra number that is modified by the determinant. One may define $f(\alpha) = ((k+1)/2)\alpha^q$, in such a way $\int_0^1 f(\alpha)\mathrm{d}\alpha = 1/2$. Working with (3) in view of $f(\alpha) = (1-\alpha)^q$, as to

$$\begin{aligned} \mathrm{Val}(T(u)) &= \frac{\int_0^1 (\underline{T}(u)(\alpha) + \overline{T}(u)(\alpha)/2))(1-\alpha)^q \mathrm{d}\alpha}{\int_0^1 (1-\alpha)^q \mathrm{d}\alpha} \\ &= \frac{\int_0^1 [2t_c + (1-\alpha)(t_r - 2t_c + t_l)](1-\alpha)^q \mathrm{d}\alpha}{2\int_0^1 (1-\alpha)^q \mathrm{d}\alpha}. \end{aligned} \tag{16}$$

One can write

$$\mathrm{Val}(T(u)) = \frac{t_c}{q+2} + \frac{(q+1)}{2(q+2)}(t_l + t_r). \tag{17}$$

Including

$$P=[\mathrm{Val}(u)-\mathrm{Val}(T(u))]^2+w\int_0^1 f(\alpha)[\underline{u}(\alpha)-\underline{T}(u)(\alpha)]^2\mathrm{d}\alpha+w\int_0^1 f(\alpha)[\overline{u}(\alpha)-\overline{T}(u)(\alpha)]^2\mathrm{d}\alpha,$$

$$P=\left(B-\frac{t_c}{q+2}+\frac{(q+1)}{2(q+2)}(t_l+t_r)\right)^2+w\int_0^1 f(\alpha)[\underline{u}(\alpha)-(t_c-(1-\alpha)(t_c-t_l))]^2\mathrm{d}\alpha \tag{18}$$

$$+w\int_0^1 f(\alpha)[\overline{u}(\alpha)-(t_c+(1-\alpha)(t_r-t_c))]^2\mathrm{d}\alpha.$$

Such that $B=\mathrm{Val}(u)$. To gain the core of $T(u)$, the authors must compute partial derivatives with respect to t_l and t_r. In the case of $f(\alpha)=\alpha$, we can make the following observation:

$$\frac{\partial P}{\partial t_l}=2\left(B-\frac{t_c}{q+2}-\frac{(q+1)(t_l+t_r)}{2(q+2)}\right)\left(\frac{-(q+1)}{2(q+2)}\right)-2w\int_0^1\alpha(\underline{u}(\alpha)-t(t_c-(1-\alpha)(t_c-t_l)))(1-\alpha)\mathrm{d}\alpha,$$

$$\frac{\partial P}{\partial t_r}=2\left(B-\frac{t_c}{q+2}-\frac{(q+1)(t_l+t_r)}{2(q+2)}\right)\left(\frac{-(q+1)}{2(q+2)}\right)-2w\int_0^1\alpha(\overline{u}(\alpha)-t(t_c+(1-\alpha)(t_r-t_c)))(1-\alpha)\mathrm{d}\alpha. \tag{19}$$

So as to lighten P, we obtain $\partial P/\partial t_l=0$, $\partial P/\partial t_r=0$ and then, $q\longrightarrow\infty$ we compute:

$$\begin{cases} -B+\dfrac{t_l+t_r}{2}-2w\displaystyle\int_0^1\alpha\left[\underline{u}(\alpha)(1-\alpha)-t_c(1-\alpha)+(t_c-t_l)(1-\alpha)^2\right]\mathrm{d}\alpha=0,\\ -B+\dfrac{t_l+t_r}{2}-2w\displaystyle\int_0^1\alpha\left[\overline{u}(\alpha)(1-\alpha)-t_c(1-\alpha)-(t_r-t_c)(1-\alpha)^2\right]\mathrm{d}\alpha=0.\end{cases} \tag{20}$$

If the above system is solved, we achieve the generalized fuzzy number $T(u)$ which is the best to u as follows:

$$\begin{cases} t_l(u)=\dfrac{1}{6+w}\left\{6B-wt_c-36\displaystyle\int_0^1\alpha\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha+(36+12w)\int_0^1\alpha\,\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\right\},\\ t_c=\mathrm{Val}(u)=\mathrm{Average}([u]_1),\\ t_r(u)=\dfrac{1}{6+w}\left\{6B-wt_c-36\displaystyle\int_0^1\alpha\,\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha+(36+12w)\int_0^1\alpha\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\right\}.\end{cases} \tag{21}$$

Now, we need to go through the subsequent theorem.

Theorem 1. *Assume that T and T' are the nearest triangular approximations of fuzzy number u, such that $T(u)$ is the nearest triangular fuzzy number tou, defined in Equation (21), and $T'(u)$ is the triangular fuzzy number which is the nearest to u that is defined in Equation (7). Suppose that $T(u)=(t_l,t_c,t_r)$ and $T'=(t_l',t_c',t_r')$, then $d(u,T(u))\le D(u,T'(u))$.*

Proof. The researchers are familiar with the function $f(\alpha)=(k+1/2)\alpha^k$ is nonnegative and increasing on $[0,1]$ with $f(0)=0$

and $\int_0^1 f(\alpha)\mathrm{d}\alpha = 1/2$. Then, we have $f(\alpha) \le 1/2 \le 1$, $\forall \alpha \in [0,1]$. Hence,

$$\begin{aligned}\left(\int_0^1 f(\alpha) D_2^2(u, T(u))\mathrm{d}\alpha\right)^{1/2} &\le \left(\int_0^1 f(\alpha) D_2^2(u, T'(u))\mathrm{d}\alpha\right)^{1/2}\\ &\le \left(\int_0^1 D_2^2(u, T'(u))\mathrm{d}\alpha\right)^{1/2},\\ d(u, T(u)) &\le D(u, T(u)).\end{aligned} \tag{22}$$

□

Theorem 2. *The best generalized fuzzy quantities to a given fuzzy quantity u exist and are unique. Now, let's go to the following proof.*

Proof. Let

$$\begin{bmatrix} \dfrac{2(q+1)^2}{(2q+4)^2} + \dfrac{2}{3}w & \dfrac{2(q+1)^2}{(2q+4)^2} \\ \dfrac{2(q+1)^2}{(2q+4)^2} & \dfrac{2(q+1)^2}{(2q+4)^2} + \dfrac{2}{3}w \end{bmatrix}. \tag{23}$$

Be the Hessian matrix for (3.11). As w, q are positive, the mentioned matrix is a positive definite matrix. Therefore, the result is a tool to minimize and isolate the result. □

Theorem 3. *The best triangular approximation is a fuzzy number.*

Proof. Let $T(u)$ be the nearest triangular approximation to a given fuzzy number u. Should $T(u)$ in (21), be a fuzzy number, the satisfied conditions for Definition 2 can properly be met. Let the parametric form of $T(u)$ be $(\underline{T}(u), \overline{T}(u))$; thus, the first and second conditions will be satisfied; $\underline{T}(u) = t_c - (1-\alpha)(t_c - t_l)$ is a bounded monotonic increasing left continuous function, and $\overline{T}(u) = t_c + (1-\alpha)(t_r - t_c)$ is a bounded monotonic decreasing left continuous function. We will attempt to prove that it will satisfy the third condition, i.e., $\underline{T}(u) < \overline{T}(u)$; therefore, $t_c - (1-\alpha)(t_c - t_l) \le t_c + (1-\alpha)(t_r - t_c)$ must hold. So, we should show that $t_l \le t_r$. We have knowledge that

$$\begin{aligned}&\underline{u}(\alpha) \le \overline{u}(\alpha) \Rightarrow \alpha\,\underline{u}(\alpha)(1-\alpha) \le \alpha\overline{u}(\alpha)(1-\alpha), \quad 0 \le \alpha \le 1\\ &\Rightarrow \int_0^1 \alpha\,\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha \le \int_0^1 \alpha\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\\ &\Rightarrow (36+12w)\int_0^1 \alpha\,\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha \le (36+12w)\int_0^1 \alpha\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha, \quad w \ge 0.\end{aligned} \tag{24}$$

So far, we have

$$-36\int_0^1 \alpha\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha \le -36\int_0^1 \alpha\,\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha, \tag{25}$$

$$\begin{aligned}6B - wt_c - 36\int_0^1 \alpha\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha &\le 6B - wt_c\\ &\quad -36\int_0^1 \alpha\,\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha.\end{aligned} \tag{26}$$

By summing up (24) and (26), the following can be worked out:

$$\begin{aligned}&6B - wt_c - 36\int_0^1 \alpha\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + (36+12w)\int_0^1 \alpha\,\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\\ &\le 6B - wt_c - 36\int_0^1 \alpha\,\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + (36+12w)\int_0^1 \alpha\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha.\end{aligned} \tag{27}$$

Multiplying the above relation by $1/6 + w$ shows the following results:

$$\begin{aligned}&\frac{1}{6+w}\left\{6B - wt_c - 36\int_0^1 \alpha\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + (36+12w)\int_0^1 \alpha\,\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\right\}\\ &\le \frac{1}{6+w}\left\{6B - wt_c - 36\int_0^1 \alpha\,\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + (36+12w)\int_0^1 \alpha\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\right\}.\end{aligned} \tag{28}$$

$t_l(u) \le t_r(u)$ are introduced. This operation completes the proof. □

4. Attribute

In this part, for creditworthiness, we regard quite a number of desirable features of the approximation suggested in the foregoing section.

Theorem 4. *If $u = (u_l, u_c, u_r)$ is a generalized fuzzy number, then $T(u) = u$.*

Proof. In view of the fact that u is a generalized fuzzy number, with (21), we get

$$t_c(u) = \mathrm{Val}(u) = \mathrm{Average}([u]_1) = u_c \Rightarrow t_c(u) = u_c. \tag{29}$$

And $B = \text{Average}([u]_0) = u_l + u_r/2$. Therefore, by (21), the following will be obtained:

$$t_r(u) = \frac{1}{6+w}\left\{6\left(\frac{u_l+u_r}{2}\right) - wu_c - 36\int_0^1 \alpha\left(u_c - (1-\alpha)(u_c-u_l)\right)(1-\alpha)\mathrm{d}\alpha + (36+12w)\int_0^1 \alpha\left(u_c + (1-\alpha)(u_r-u_c)\right)(1-\alpha)\mathrm{d}\alpha\right\},$$

$$\begin{aligned} t_r(u) &= \frac{1}{6+w}\left\{3u_l + 3u_r - wt_c - 36\left(\frac{u_c}{12}+\frac{u_l}{12}\right) + (36+12w)\left(\frac{u_c}{12}+\frac{u_r}{12}\right)\right\} \\ &= \frac{1}{6+w}\{3u_l + 3u_r - wu_c - 3u_c - 3u_l + 3u_c + 3u_r + wu_c + wu_r\} \\ &= \frac{1}{6+w}\{(6+w)u_r\} \\ &= u_r, \end{aligned} \tag{30}$$

and

$$t_l(u) = \frac{1}{6+w}\left\{6\left(\frac{u_l+u_r}{2}\right) - wu_c - 36\int_0^1 \alpha\left(u_c + (1-\alpha)(u_r-u_c)\right)(1-\alpha)\mathrm{d}\alpha + (36+12w)\int_0^1 \alpha\left(u_c - (1-\alpha)(u_c-u_l)\right)(1-\alpha)\mathrm{d}\alpha\right\}. \tag{31}$$

In a similar way, by simplification, we get $t_l(u) = u_l$. We then obtain $t_r(u) = u_r$ and $t_l(u) = u_l$ which finalize the proof. □

Theorem 5. *The best generalized triangular approximation is invariant to translation, such as* $T(u+z) = T(u) + z$ *for all* $z \in R$ *and* $u \in$ E.

Proof. Assume z represents an arbitrary crisp number and u specifies a fuzzy quantity. By (21), we realize

$$t_c(u+z) = \text{Average}([u]_1 + z) = \text{Average}([u]_1) + z = t_c + z, \tag{32}$$

and

$$\begin{aligned} B' &= \text{Average}([u]_0 + z) = \text{Average}([u]_0) + z = B + z, \\ t_l(u+z) &= \frac{1}{6+w}\left\{6(B+z) - wt_c(u+z) - 36\int_0^1 \alpha(\overline{u}(\alpha)+z)(1-\alpha)\mathrm{d}\alpha + (36+12w)\int_0^1 \alpha(\underline{u}(\alpha)+z)(1-\alpha)\mathrm{d}\alpha\right\} \\ &= \frac{1}{6+w}\left\{6B + 6z - wt_c - wz - 36\int_0^1 \alpha\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha - 6z + (36+12w)\int_0^1 \alpha\,\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + 6z + 2wz\right\} \\ &= \frac{1}{6+w}\left\{6B - wt_c - 36\int_0^1 \alpha\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + (36+12w)\int_0^1 \alpha\,\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + (6+w)z\right\} \\ &= \frac{(6+w)z}{6+w}\left\{6B - wt_c - 36\int_0^1 \alpha\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + (36+12w)\int_0^1 \alpha\,\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\right\} \\ &= t_l(u) + z. \end{aligned} \tag{33}$$

It is obvious from (21) that similar computations ending in $t_r(u+z) = t_r(u) + z$ will be accomplished. This should imply$T(u+z) = T(u) + z$, which proves the translation invariance. □

Theorem 6. *The best generalized approximation is a scale invariant. There is an excellent case in point:* $T(\mu u) = \mu T(u), \quad \forall \mu \in \Re\backslash\{0\}, u \in E$.

Proof. Let μ typify a nonzero crisp value. By (21), the authors gain $t_c(\mu u) = \text{Average}(\mu[u]_1) = \mu\text{Average}([u]_1) = \mu t_c(u)$ and $B' = \text{Average}(\mu[u]_0) = \mu B$. Then,

$$t_l(\mu u) = \frac{1}{6+w}\left\{6(\mu B) - wt_c(\mu u) - 36\int_0^1 \alpha(\mu\overline{u}(\alpha))(1-\alpha)\mathrm{d}\alpha + (36+12w)\int_0^1 \alpha(\mu\underline{u}(\alpha))(1-\alpha)\mathrm{d}\alpha\right\} = \mu t_l(u). \quad (34)$$

One can pinpoint that $t_r(\mu u) = \mu t_r(u)$. As a result, we earn $T(\mu u) = \mu T(u)$, which consummate the proof. □

Theorem 7. *The best-generalized approximation is monotonic if u, v are symmetric and $Average([u]_1) = Average([v]_1)$.*

Proof. Let $u = (u_l, u_c, u_r)$ and $v = (v_l, v_c, v_r)$ denote two arbitrary fuzzy numbers. Furthermore, let $T_u = (t_l, t_c, t_r)$ and $T_v = (t'_l, t'_c, t'_r)$ be two triangular fuzzy numbers which should be computed using u and v, sequentially. We assume that $u\subseteq v$; accordingly, u, v believed to be symmetric; hence, $\underline{v}(\alpha) \leq \underline{u}(\alpha)$ and $\overline{u}(\alpha) \leq \overline{v}(\alpha)$ for each and every $\alpha \in (0,1]$. Since $\text{Average}([u]_1) = \text{Average}([v]_1)$, proved by (21), we observe $t_c(u) = t_c(v)$, i.e., $t_c = t'_c$. Also, $\text{Average}([u]_0) = \text{Average}([v]_0)$, i.e., $B = B'$. Consequently, by (9), the following will be worked out:

$$\begin{aligned} t_l &= \frac{1}{6+w}\left\{6B - wt_c - 36\int_0^1 \alpha\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + (36+12w)\int_0^1 \alpha\,\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\right\} \\ &\geq \frac{1}{6+w}\left\{6B' - wt'_{c\,c} - 36\int_0^1 \alpha\overline{v}(\alpha)(1-\alpha)\mathrm{d}\alpha + (36+12w)\int_0^1 \alpha\,\underline{v}(\alpha)(1-\alpha)\mathrm{d}\alpha\right\} \\ &= t'_l. \end{aligned} \quad (35)$$

In this regard, we are allowed to present $t_r \leq t'_r$, which leads to $T(u)\subseteq T(v)$, and the criterion of monotony holds. □

5. Numerical Example

Example 1. Offer u as a fuzzy quantity of the following parametric applying $u(\alpha) = (2\alpha - 2, 1 - \sqrt{\alpha})$, $\alpha \in (0,1]$. Now, designate $w = 10$, then having (21) and (7), we demonstrate Table 1.

The accomplished results are exhibited in Figure 1.

Table 1: Example 1.

t_l	t_c	t_r	Equation
−1.980434783	0	0.7195652174	(7)
−1.930357143	0	0.698214285	(21)

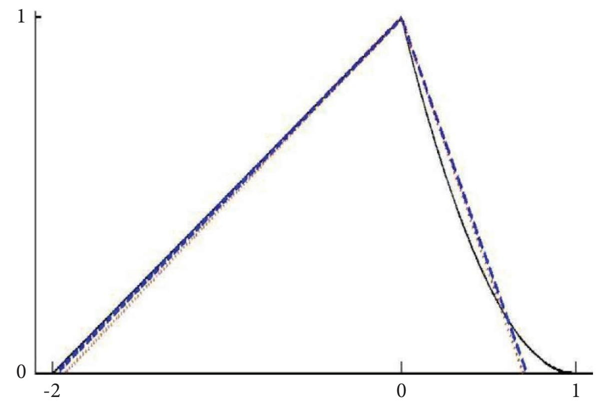

Figure 1: The best generalized fuzzy quantity for $w = 10$, solid curve: u; dashed curve: equation (7); dotted curve: equation (21).

Table 2: Results of Example 2.

t_l	t_c	t_r	Equation
−0.7	0	0.7	(7)
−0.628571428	0	0.628571428	(21)

Example 2. Assume that u enumerate a fuzzy quantity of the following parametric subject to $u(\alpha) = (\alpha^{1/2} - 1, 1 - \alpha^{1/2})$, $\alpha \in (0,1]$. By (21) and (7), considering setting $w = 10$, we observe Table 2.

The attained results are pictured in Figure 2.

Another example will be helpful.

Example 3. Assume that u is a fuzzy quantity of the next parametric; reconcile to $u(\alpha) = (1 + \alpha^{1/2}, 30 - 27\alpha^{1/2})$, $\alpha \in (0,1]$. By (21) and (7), and setting $w = 10$, we accomplish Table 3. (See Figure 3).

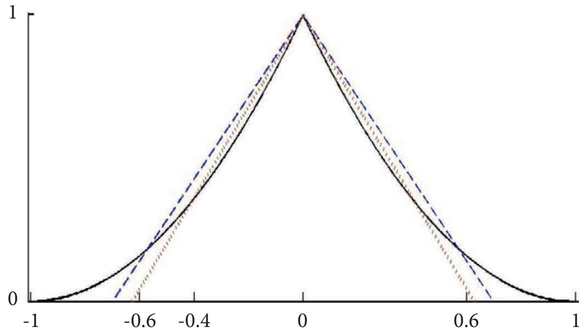

Figure 2: The best generalized fuzzy quantity for $w = 10$, solid curve: u; dashed curve: equation (7); dotted curve: equation (21).

Table 3: Results of Example 3.

t_l	t_c	t_r	Equation
1.558695652	2.5	22.65869565	(7)
2.682142857	2.5	22.28214286	(21)

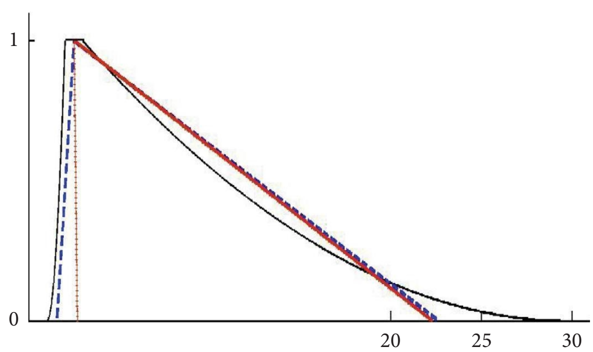

Figure 3: The best generalized fuzzy quantity for $w = 10$, solid curve: u; dashed curve: equation (7); dotted curve: equation (21).

6. Conclusion

This paper attempted to suggest an advanced method for the α-scaled generalized approximation of general fuzzy quantities. The given procedure yields generalized fuzzy quantities that leave no doubts about their being straightforward and unequivocal. Then, feasible properties of the α-scaled generalized approximation of a fuzzy quantity were ferreted out.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] M. Delgado, M. A. Vila, and W. Voxman, "On a canonical representation of fuzzy numbers," *Fuzzy Sets and Systems*, vol. 93, no. 1, pp. 125–135, 1998.

[2] M. Detyniecki and R. R. Yager, "Ranking fuzzy numbers using α-weighted valuations," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 8, no. 5, pp. 573–591, 2000.

[3] P. Grzegorzewski, "Nearest interval approximation of a fuzzy number," *Fuzzy Sets and Systems*, vol. 130, no. 3, pp. 321–330, 2002.

[4] M. Ma, A. Kandel, and M. Friedman, "A new approach for defuzzification," *Fuzzy Sets and Systems*, vol. 111, no. 3, pp. 351–356, 2000.

[5] R. R. Yager and D. Filev, "On ranking fuzzy numbers using valuations," *International Journal of Intelligent Systems*, vol. 14, no. 12, pp. 1249–1268, 1999.

[6] R. Yager and D. P. Filev, *On the Instantiation of Possibility Distributions*, Machine Intelligence Institute, Berkeley, CA, USA, 1998.

[7] W. L. Hung and J. W. Wu, "Correlation of intuitionistic fuzzy sets by centroid method," *Information Sciences*, vol. 144, no. 1–4, pp. 219–225, 2002.

[8] S. Chanas, "On the interval approximation of a fuzzy number," *Fuzzy Sets and Systems*, vol. 122, no. 2, pp. 353–356, 2001.

[9] S. Abbasbandy and B. Asady, "The nearest trapezoidal fuzzy number to a fuzzy quantity," *Applied Mathematics and Computation*, vol. 156, no. 2, pp. 381–386, 2004.

[10] S. Abbasbandy, E. Ahmady, and N. Ahmady, "Triangular approximations of fuzzy numbers using α-weighted valuations," *Soft Computing*, vol. 14, no. 1, pp. 71–79, 2010.