arXiv:2312.13609v3 [math.FA] 27 Sep 2024

Toeplitz Operators Defined between Köthe Spaces

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Abstract. The aim of this paper is to define Toeplitz operators between Köthe spaces, especially power series spaces. We determine the conditions for continuity and compactness of these operators. We define the concept of S-tameness of a family of continuous operators. We construct some conditions on S-tameness for the families consisting of Toeplitz operators.

Mathematics Subject Classification (2010). 46A45, 47A05, 47B35, 46A61.

Keywords. Toeplitz Matrix, Köthe Spaces, Compact Operators.

1. Introduction

The theory of Toeplitz operators defined on a Hilbert space such as the Hardy and the Bergman spaces is well-studied. The significance of this theory lies in the connection between operator theory and function spaces. The matrix of a Toeplitz operator defined on Hardy space of unit disk $H^2(\mathbb{D})$ is a Toeplitz matrix. Moreover, any bounded operator on $H^2(\mathbb{D})$ whose associated matrix is a Toeplitz matrix is a Toeplitz operator. We direct the reader to [8] for more information about Toeplitz operators defined on $H^2(\mathbb{D})$. In recent years, Toeplitz operators, whose "associated" matrix is Toeplitz, are defined for more general topological vector spaces. For instance, in [3], Domański and Jasiczak developed the analogous theory for the space of $\mathcal{A}(\mathbb{R})$ real analytic functions on the real line. This space is not a Banach space, even not a metrizable space. In [6], Jasiczak introduced and characterized the class of Toeplitz operators on the Fréchet space of all entire functions $\mathcal{O}(\mathbb{C})$.

In [6], Jasiczak defined a continuous linear operator on $\mathcal{O}(\mathbb{C})$ as a Toeplitz operator if its matrix is a Toeplitz matrix. The matrix of an operator is defined with respect to the Schauder basis $(z^n)_{n\in N_0}$. In this case, the symbol of a Toeplitz operator comes from the space $\mathcal{O}(\mathbb{C})\oplus(\mathcal{O}(\mathbb{C}))'_b$ here $(\mathcal{O}(\mathbb{C}))'_b$ is the strong dual of $\mathcal{O}(\mathbb{C})$. The space of entire functions $\mathcal{O}(\mathbb{C})$ is isomorphic to a power series space of infinite type $\Lambda_\infty(n)$. By taking inspiration from Jasiczak paper [6], we will define Toeplitz operators on more general power series spaces of finite or infinite type. Similarly, we will show that the symbol space is $\Lambda_r(\alpha) \oplus (\Lambda_r(\alpha))'$, for $r = 1, \infty$. We will also search the compactness of these operators and mention the conditions for the tameness of the families consisting of Toeplitz operators.

2. Preliminaries

In this section, we provide some fundamental facts and definitions essential for the subsequent discussions. We will use the standard terminology and notation of [9].

A complete Hausdorff locally convex space E whose topology defined by countable fundamental system of seminorms $(\|\cdot\|_k)_{k\in\mathbb{N}}$ is called a Fréchet space. A matrix $(a_{n,k})_{k,n\in\mathbb{N}}$ of non-negative numbers is called a Köthe matrix if it is satisfies the following conditions:

1. For each $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ with $a_{n,k} > 0$.
2. $a_{n,k} \leqslant a_{n,k+1}$ for each $n, k \in \mathbb{N}$.

For a Köthe matrix $(a_{n,k})_{n,k\in\mathbb{N}}$,

$$K(a_{n,k}) = \left\{ x = (x_n)_{n\in\mathbb{N}} : \ \|x\|_k := \sum_{n=1}^{\infty} |x_n| a_{n,k} < \infty \quad \text{for all} \quad k \in \mathbb{N} \right\}$$

is called a Köthe space. Every Köthe space is a Fréchet space. From Proposition 27.13 of [9], the dual space of a Köthe space is

$$(K(a_{n,k}))' = \left\{ y = (y_n)_{n\in\mathbb{N}} \;\middle|\; \sup_{n\in\mathbb{N}} |y_n a_{n,k}^{-1}| < +\infty \text{ for some } k \in \mathbb{N} \right\}.$$

By Grothendieck-Pietsch Criteria [9][Theorem 28.15] a Köthe space $K(a_{n,k})$ is nuclear if and only if for every $k \in \mathbb{N}$, there exists a $l > k$ so that

$$\sum_{n=1}^{\infty} \frac{a_{n,k}}{a_{n,l}} < \infty.$$

For a nuclear Köthe spaces, $\|x\|_k = \sup_{n\in\mathbb{N}} |x_n| a_{n,k}$, $k \in \mathbb{N}$ forms an equivalent system of seminorms to the fundamental system of seminorms $\|x\|_k = \sum_{n=1}^{\infty} |x_n| a_{n,k}$, $k \in \mathbb{N}$, see Proposition 28.16 of [9].

Dynin-Mitiagin Theorem [9][Theorem 28.12] states that if a nuclear Fréchet space E with the sequence of seminorms $(\|f_n\|)_{n\in\mathbb{N}}$ has a Schauder basis $(f_n)_{n\in\mathbb{N}}$, then it is canonically isomorphic to a nuclear Köthe space defined by the matrix $(\|f_n\|_k)_{n,k\in\mathbb{N}}$. So, nuclear Köthe spaces hold a significant place in theory of nuclear Fréchet space.

Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a non-negative increasing sequence with $\lim_{n\to\infty} \alpha_n = \infty$. A power series space of finite type is defined by

$$\Lambda_1(\alpha) := \left\{ x = (x_n)_{n\in\mathbb{N}} : \ \|x\|_k := \sum_{n=1}^{\infty} |x_n| \, e^{-\frac{1}{k}\alpha_n} < \infty \text{ for all } k \in \mathbb{N} \right\}$$

and a power series space of infinite type is defined by

$$\Lambda_\infty(\alpha) := \left\{ x = (x_n)_{n\in\mathbb{N}} : \|x\|_k := \sum_{n=1}^{\infty} |x_n| e^{k\alpha_n} < \infty \text{ for all } k \in \mathbb{N} \right\}.$$

Power series spaces form an important family of Köthe spaces and they contain the spaces of holomorphic functions on $\mathbb{C}^d$ and $\mathbb{D}^d$,

$$\mathcal{O}(\mathbb{C}^d) \cong \Lambda_\infty(n^{\frac{1}{d}}) \qquad \text{and} \qquad \mathcal{O}(\mathbb{D}^d) \cong \Lambda_1(n^{\frac{1}{d}})$$

where $\mathbb{D}$ is the unit disk in $\mathbb{C}$ and $d \in \mathbb{N}$.

Let E and F be Fréchet spaces. A linear map $T : E \to F$ is called continuous if for every $k \in \mathbb{N}$ there exists $p \in \mathbb{N}$ and $C_k > 0$ such that

$$\|Tx\|_k \leqslant C_k \|x\|_p$$

for all $x \in E$. A linear map $T : E \to F$ is called compact if $T(U)$ is precompact in F where U is a neighborhood of zero of E.

In this paper, we fixed the symbol e_n to denote the sequence

$$(0, 0, \dots, 0, 1, 0, \dots)$$

where 1 is in the nth place and 0 is in the others.

We will use the following Lemma to determine the continuity and compactness of operators defined between Köthe spaces.

Lemma 2.1. *Let $K(a_{n,k})$ and $K(b_{n,k})$ be Köthe spaces.*

a. *$T : K(a_{n,k}) \to K(b_{n,k})$ is a linear continuous operator if and only if for each k there exists m such that*

$$\sup_{n\in\mathbb{N}} \frac{\|Te_n\|_k}{\|e_n\|_m} < \infty.$$

b. *If $K(b_{n,k})$ is Montel, then $T : K(a_{n,k}) \to K(b_{n,k})$ is a compact operator if and only if there exists m such that for all k*

$$\sup_{n\in\mathbb{N}} \frac{\|Te_n\|_k}{\|e_n\|_m} < \infty.$$

Proof. Lemma 2.1 of [2]. □

We want to note that a Fréchet space E is Montel if each bounded set in E is relatively compact and every power series space is Montel, see Theorem 27.9 of [9].

In the next proposition, it will be shown that the continuity condition is sufficient to ensure that linear operators defined only on the basis elements are well-defined.

Proposition 2.2. *Let $K(a_{n,k})$, $K(b_{n,k})$ be Köthe spaces and $(a_n)_{n\in\mathbb{N}} \in K(b_{n,k})$ be a sequence. Let us define a linear map $T : K(a_{n,k}) \to K(b_{n,k})$ such as*

$$Te_n = a_n \qquad \textit{and} \qquad Tx = \sum_{n=1}^{\infty} x_n Te_n$$

for every $n \in \mathbb{N}$ and $x = \sum_{n=1}^{\infty} x_n e_n$. If the continuity condition

$$\forall k \in \mathbb{N} \qquad \exists m \in \mathbb{N} \qquad \sup_{n\in\mathbb{N}} \frac{\|Te_n\|_k}{\|e_n\|_m} < \infty$$

holds, then T is well-defined and continuous operator.

Proof. Let $x = \sum_{n=1}^{\infty} x_n e_n$ be an arbitrary element of $K(a_{n,k})$ and assume that for every $k \in \mathbb{N}$ there exists a $m \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} \frac{\|Te_n\|_k}{\|e_n\|_m} < \infty.$$

Now we define the sequence

$$y_N = \sum_{n=1}^{N} T(x_n e_n) = \sum_{n=1}^{N} x_n T e_n$$

for every $N \in \mathbb{N}$. Then for every $k \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and $C > 0$ such that for sufficiently large $N, M \in \mathbb{N}$, $N > M$,

$$\|y_N - y_M\|_k = \| \sum_{l=M+1}^{N} x_l T(e_l)\|_k \leqslant \sum_{l=M+1}^{N} |x_l| \|e_l\|_m \frac{\|Te_l\|_k}{\|e_l\|_m} \leqslant C \sum_{l=M+1}^{N} |x_l| \|e_l\|_m$$

which is arbitrarily small, since $\|x\|_m = \sum_{n=1}^{\infty} |x_n| \|e_n\|_m < \infty$. This says that the sequence $(y_N)_{N\in\mathbb{N}} \subseteq K(b_{k,n})$ is a Cauchy sequence and therefore has a limit. Let us say the $\lim_{N\to\infty} y_N = y$. Hence, we have

$$y = \lim_{N\to\infty} y_N = \sum_{n=1}^{\infty} x_n T(e_n) = T(x) \in K(b_{k,n}).$$

Therefore T is well-defined from $K(a_{k,n})$ to $K(b_{k,n})$ and T is continuous from Lemma 2.1. □

A grading on a Fréchet space E is a sequence of seminorms $\{\|\cdot\|_n : n \in \mathbb{N}\}$ which are increasing, that is,

$$\|x\|_1 \leqslant \|x\|_2 \leqslant \|x\|_3 \leqslant \dots$$

for each $x \in E$ and which define the topology. Every Fréchet space admits a grading. A graded Fréchet space is Fréchet space with a choice of grading. For more information, see [5]. In this paper, unless stated otherwise, we will assume that all Fréchet spaces are graded Fréchet spaces.

A pair of graded Fréchet spaces (E, F) is called tame provided that there exists an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$, such that for every continuous linear operator T from E to F, there exists an N and $C > 0$ such that

$$\|Tx\|_n \leqslant C\|x\|_{\sigma(n)} \qquad \forall x \in E \text{ and } n \geqslant N.$$

A Fréchet space E is called tame if (E, E) is tame. Tameness gives us a kind of control on the continuity of the operators. In [4], Dubinsky and Vogt used this notion to find a basis in complemented subspaces of some infinite type power series spaces. We refer to [1] and [7] for the other studies about tameness.

In this paper, instead of all operators defined on a Fréchet space, we will restrict ourselves to a subfamily of operators. With this aim, we will give the following definition.

Definition 2.3. *Let $S : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. A family of linear continuous operators $\mathcal{A} \subseteq L(E, F)$ is called S-tame if for every operator $T \in \mathcal{A}$ there exist $k_0 \in \mathbb{N}$ and $C > 0$ such that*

$$\|Tx\|_k \leqslant C\|x\|_{S(k)} \qquad \forall x \in E, k \geqslant k_0.$$

S-tameness of a family can be given by considering only elements of bases similar to Lemma 2.1.

Lemma 2.4. *Let $K(a_{n,k})$ and $K(b_{n,k})$ be Köthe spaces and $S : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. A family of linear continuous operators $\mathcal{A} \subseteq L(K(a_{n,k}), K(b_{n,k}))$ is S-tame if and only if for every operator $T \in \mathcal{A}$ there exist $k_0 \in \mathbb{N}$ and $C > 0$ such that*

$$\|Te_n\|_k \leqslant C\|e_n\|_{S(k)} \qquad \forall n \in \mathbb{N}, k \geqslant k_0.$$

Proof. Let us assume that for every $T \in \mathcal{A}$ there exist a $k_0 \in \mathbb{N}$ and $C > 0$ such that

$$\|Te_n\|_k \leqslant C\|e_n\|_{S(k)} \qquad \forall n \in \mathbb{N}, k \geqslant k_0.$$

Then for every $x = \sum_{n=1}^{\infty} x_n e_n$, we can write

$$\begin{aligned}\|Tx\|_k &= \left\| \sum_{n=1}^{\infty} x_n T(e_n) \right\|_k \leqslant \sum_{n=1}^{\infty} |x_n| \|Te_n\|_k \leqslant C \sum_{n=1}^{\infty} |x_n| \|e_n\|_{S(k)} \\ &= C \sum_{n=1}^{\infty} |x_n| a_{n,S(k)} = C\|x\|_{S(k)}.\end{aligned}$$

This says that $\mathcal{A}$ is S-tame. The other direction is straightforward. □

In this paper, we will call an operator which is defined between Köthe spaces as a Toeplitz operator if its matrix is a Toeplitz matrix defined with respect to the Schauder basis $(e_n)_{n\in\mathbb{N}}$. In sections 3 and 4, we will concentrate on lower triangular Toeplitz matrices and upper triangular Toeplitz matrices, respectively. We will determine the continuity and compactness of the operators whose associated matrix is related to these matrices. By collecting the results obtained in sections 3 and 4, we share the results about the continuity and compactness of a Toeplitz operator defined between power series spaces in section 5.

3. Operators Defined by A Lower Triangular Toeplitz Matrix

Let $\theta = (\theta_n)_{n\in\mathbb{N}}$ be any sequence. The lower triangular Toeplitz matrix defined by θ is

$$\begin{pmatrix} \theta_0 & 0 & 0 & 0 & \cdots \\ \theta_1 & \theta_0 & 0 & 0 & \cdots \\ \theta_2 & \theta_1 & \theta_0 & 0 & \cdots \\ \theta_3 & \theta_2 & \theta_1 & \theta_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We want to define an operator $\widehat{T}_\theta : K(a_{n,k}) \to K(b_{n,k})$ by taking $\widehat{T}_\theta e_n$ as the nth column of the above matrix, that is,

$$\widehat{T}_\theta e_n = (0, \cdots, 0, \theta_0, \theta_1, \theta_2, \cdots) = \sum_{j=n}^{\infty} \theta_{j-n} e_j$$

provided that $\widehat{T}_\theta e_n \in K(b_{n,k})$ for every $n \in \mathbb{N}$. Therefore, for every $x = \sum_{n=1}^{\infty} x_n e_n \in K(a_{n,k})$, the operator $\widehat{T}_\theta$ is given by

$$\widehat{T}_\theta x = \sum_{n=1}^{\infty} x_n \widehat{T}_\theta e_n.$$

In fact, we can not guarantee that the operator $\widehat{T}_\theta$ is correctly defined between two general Köthe spaces $K(a_{n,k})$ and $K(b_{n,k})$, because we do not know if the series $\sum_{n=1}^{\infty} x_n \widehat{T}_\theta e_n$ converges in $K(b_{n,k})$ for every $x \in K(a_{n,k})$ in this general case. Below, we will examine the cases where this operator can be correctly defined, and in these cases, we will analyze the continuity and compactness of this operator. Throughout this paper, we will assume that the sequences θ satisfy the following condition

$$\forall n \in \mathbb{N} \quad \exists s > n \qquad \theta_s \neq 0.$$

Otherwise, θ produces a finite rank operator, which is obviously continuous and compact.

The idea of Proposition 3.1 is the same as [10, Theorem 2.1]. Therein one can also find a generalized form of Theorem 3.3.

Proposition 3.1. *Let $K(a_{n,k})$, $K(b_{n,k})$ be Köthe spaces and assume that θ be a sequence and $s_0 = \min\{t : \theta_t \neq 0\}$. If $\widehat{T}_\theta : K(a_{n,k}) \to K(b_{n,k})$ is a continuous operator, then $\theta \in K(b_{n,k})$ and the following holds*

$$\forall k \in \mathbb{N} \quad \exists m \in \mathbb{N}, C > 0 \qquad b_{n+s_0,k} \leqslant C a_{n,m} \qquad \forall n \in \mathbb{N}.$$

Proof. Let $\widehat{T}_\theta : K(a_{n,k}) \to K(b_{n,k})$ be a continuous operator. For every $n \in \mathbb{N}$,

$$\widehat{T}_\theta e_n = (0, \ldots, 0, \theta_0, \theta_1, \theta_2, \ldots) = \sum_{j=n}^{\infty} \theta_{j-n} e_j \in K(b_{n,k}).$$

Since $\widehat{T}_\theta e_1 \in K(b_{n,k})$, this gives us $\theta \in K(b_{n,k})$. By Lemma 2.1, for all $k \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C_1 > 0$ such that

$$\|\widehat{T}_\theta e_n\|_k = \sum_{j=n}^{\infty} |\theta_{j-n}| b_{j,k} \leqslant C_1 \|e_n\|_m = C_1 a_{n,m} \qquad \forall n \in \mathbb{N}.$$

Then, for all n and $j \geqslant n$, we have $|\theta_{j-n}| b_{j,k} \leqslant C_1 a_{n,m}$. Hence we can write that $b_{n+s_0,k} \leqslant C_2 a_{n,m}$ for some $C_2 > 0$. This completes the proof. □

For compact $\widehat{T}_\theta : K(a_{n,k}) \to K(b_{n,k})$ operators, the relationship between Köthe matrices $(a_{n,k})_{n,k\in\mathbb{N}}$ and $(b_{n,k})_{n,k\in\mathbb{N}}$ is as follows:

Proposition 3.2. *Let $K(a_{n,k})$ and $K(b_{n,k})$ be Köthe spaces such that $K(b_{n,k})$ is Montel and assume that $\theta \in K(b_{n,k})$ and $s_0 = \min\{t : \theta_t \neq 0\}$. If $\widehat{T}_\theta : K(a_{n,k}) \to K(b_{n,k})$ is a compact operator, then the following holds*

$$\exists m \in \mathbb{N} \quad \forall k \in \mathbb{N} \quad \exists C > 0 \qquad b_{n+s_0,k} \leqslant C a_{n,m} \qquad \forall n \in \mathbb{N}.$$

Proof. The proof is similar to the proof of Proposition 3.1. □

The converse of Proposition 3.1 is true when $K(b_{n,k})$ is a power series space of finite type.

Theorem 3.3. *Let $K(a_{n,k})$ be a Köthe space, $\Lambda_1(\beta)$ be a power series space of finite type and assume that θ be a sequence and $s_0 = \min\{t : \theta_t \neq 0\}$. $\widehat{T}_\theta : K(a_{n,k}) \to \Lambda_1(\beta)$ is well-defined and continuous if and only if $\theta \in \Lambda_1(\beta)$ and the following condition holds:*

$$\forall k \in \mathbb{N} \quad \exists m \in \mathbb{N}, C > 0 \qquad e^{-\frac{1}{k}\beta_{n+s_0}} \leqslant C a_{n,m} \qquad \forall n \in \mathbb{N}. \quad (3.1)$$

Proof. If the operator $\widehat{T}_\theta$ is continuous, Proposition 3.1 gives us that $\theta \in \Lambda_1(\beta)$ and (3.1) holds. For the other direction, let us assume $\theta \in \Lambda_1(\beta)$. By using the condition (3.1) we have the following: for every $k \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and $C_1, C_2 > 0$ such that

$$\begin{aligned}
\|\widehat{T}_\theta e_n\|_k \leqslant \|\widehat{T}_\theta e_n\|_{2k} &= \sum_{j=n}^{\infty} |\theta_{j-n}| e^{-\frac{1}{2k}\beta_j} \\
&= \sum_{j=n+s_0}^{\infty} |\theta_{j-n}| e^{-\frac{1}{2k}\beta_j} e^{\frac{1}{k^2+2k}\beta_{n+s_0}} e^{-\frac{1}{k^2+2k}\beta_{n+s_0}} \\
&\leqslant C_1 a_{n,m} \sum_{j=n+s_0}^{\infty} |\theta_{j-n}| e^{-\frac{1}{2k}\beta_j} e^{\frac{1}{k^2+2k}\beta_{n+s_0}} \\
&= C_1 a_{n,m} \sum_{j=n+s_0}^{\infty} |\theta_{j-n}| e^{-\frac{1}{2(k+2)}\beta_j} e^{-\frac{1}{k^2+2k}(\beta_j - \beta_{n+s_0})} \\
&\leqslant C_1 a_{n,m} \sum_{j=n+s_0}^{\infty} |\theta_{j-n}| e^{-\frac{1}{2(k+2)}\beta_j} = C_1 \|\theta\|_{2(k+2)} a_{n,m} = C_2 a_{n,m}
\end{aligned}$$

for every $n \in \mathbb{N}$. In the second line of the inequalites we use the fact that

$$\frac{1}{k^2+2k} = \frac{1}{2k} - \frac{1}{2(k+2)}$$

for every $k \in \mathbb{N}$. Therefore, $\widehat{T}_\theta e_n \in \Lambda_1(\beta)$ for every $n \in \mathbb{N}$ and for every $k \in \mathbb{N}$ there exist $m \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} \frac{\|\widehat{T}_\theta e_n\|_k}{\|e_n\|_m} < \infty,$$

that is, $\widehat{T}_\theta : K(a_{n,k}) \to \Lambda_1(\beta)$ is well defined and continuous by Proposition 2.2. □

We can characterize the compactness of the operators $\widehat{T}_\theta : K(a_{n,k}) \to \Lambda_1(\beta)$ as the follows.

Theorem 3.4. *Let $K(a_{n,k})$ be a Köthe space, $\Lambda_1(\beta)$ be a power series space of finite type and assume that θ be a sequence and $s_0 = \min\{t : \theta_t \neq 0\}$. $\widehat{T}_\theta : K(a_{n,k}) \to \Lambda_1(\beta)$ is compact if and only if $\theta \in \Lambda_1(\beta)$ and the following condition holds:*

$$\exists m \in \mathbb{N} \quad \forall k \in \mathbb{N} \quad \exists C > 0 \qquad e^{-\frac{1}{k}\beta_{n+s_0}} \leqslant C a_{n,m} \qquad \forall n \in \mathbb{N}. \quad (3.2)$$

Proof. If $\widehat{T}_\theta$ is compact, then $\widehat{T}_\theta$ is continuous and $\theta \in \Lambda_1(\beta)$ from Proposition 3.1. Proposition 3.2 says that the condition (3.2) holds. For the other direction, assume that $\theta \in \Lambda_1(\beta)$ and the condition (3.2) holds. By the same calculation of the proof of Theorem 3.3, we can write that there exists an $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$

$$\sup_{n\in\mathbb{N}} \frac{\|\widehat{T}_\theta e_n\|_k}{\|e_n\|_m} < \infty.$$

From Lemma 2.1, we can say that $\widehat{T}_\theta : K(a_{n,k}) \to \Lambda_1(\beta)$ is compact. □

Proposition 3.5. *Let $K(a_{n,k})$ be a Köthe space and $\Lambda_1(\beta)$ be a power series space of finite type. Assume there exists a non-decreasing function $S : \mathbb{N} \to \mathbb{N}$ such that the following condition holds:*

$$\forall k \in \mathbb{N} \quad \exists C > 0 \qquad e^{-\frac{1}{k}\beta_n} \leqslant C a_{n,S(k)} \qquad \forall n \in \mathbb{N}. \quad (3.3)$$

Then, the family of the operators $\widehat{T}_\theta : K(a_{n,k}) \to \Lambda_1(\beta)$ for $\theta \in \Lambda_1(\beta)$, that is,

$$\widehat{\mathcal{A}}_1 = \{\widehat{T}_\theta : \theta \in \Lambda_1(\beta)\}$$

is $\tilde{S}$-tame, where $\tilde{S}(k) = S(2k)$ for every $k \in \mathbb{N}$.

Proof. Let $\theta \in \Lambda_1(\beta)$. By using (3.3), for every $k \in \mathbb{N}$, there exist $C_1, C_2 > 0$ such that

$$\begin{aligned}
\|\widehat{T}_\theta e_n\|_k &= \sum_{j=n}^{\infty} |\theta_{j-n}| e^{-\frac{2}{2k}\beta_j} = \sum_{j=n}^{\infty} |\theta_{j-n}| e^{-\frac{1}{2k}\beta_j} e^{-\frac{1}{2k}\beta_j} \\
&\leqslant \left(\sum_{j=n}^{\infty} |\theta_{j-n}| e^{-\frac{1}{2k}\beta_j} \right) e^{-\frac{1}{2k}\beta_n} \leqslant C_1 \left(\sum_{j=n}^{\infty} |\theta_{j-n}| e^{-\frac{1}{2k}\beta_j} \right) a_{n,S(2k)} \\
&\leqslant C_1 \left(\sum_{j=n}^{\infty} |\theta_{j-n}| e^{-\frac{1}{2k}\beta_{j-n+1}} \right) \|e_n\|_{S(2k)} \leqslant C_1 \|\theta\|_{2k} \|e_n\|_{\tilde{S}(k)} \leqslant C_2 \|e_n\|_{\tilde{S}(k)}.
\end{aligned}$$

Therefore, for every $\theta \in \Lambda_1(\beta)$ and for all $k \in \mathbb{N}$, there exists $C > 0$ so that we write

$$\|\widehat{T}_\theta e_n\|_k \leqslant C \|e_n\|_{\tilde{S}(k)},$$

that is, the family $\widehat{\mathcal{A}}_1$ is $\tilde{S}$-tame. □

Now we turn our attention to the power series space of infinite type. For the converse of Proposition 3.1, we need the stability condition on the sequence β when $K(b_{n,k})$ is power series space of infinite type $\Lambda_\infty(\beta)$. A sequence β is called stable if

$$\sup_{n \in \mathbb{N}} \frac{\beta_{2n}}{\beta_n} < \infty.$$

Theorem 3.6. *Let $\beta = (\beta_n)_{n\in\mathbb{N}}$ be a stable sequence. $\widehat{T}_\theta : K(a_{n,k}) \to \Lambda_\infty(\beta)$ is well-defined and continuous if and only if $\theta \in \Lambda_\infty(\beta)$ and the following condition holds:*

$$\forall k \in \mathbb{N} \quad \exists m \in \mathbb{N}, C > 0 \qquad e^{k\beta_n} \leqslant C a_{n,m} \qquad \forall n \in \mathbb{N}. \quad (3.4)$$

Proof. Let assume θ be a sequence and $s_0 = \min\{t : \theta_t \neq 0\}$. If the operator $\widehat{T}_\theta$ is continuous, Proposition 3.1 gives us that $\theta \in \Lambda_\infty(\beta)$ and for every $k \in \mathbb{N}$, there exist $m \in \mathbb{N}$, $C > 0$ such that

$$e^{k\beta_{n+s_0}} \leqslant C a_{n,m}$$

and then we have

$$e^{k\beta_n} \leqslant C a_{n,m}$$

for every $n \in \mathbb{N}$. This says that (3.4) holds since β is an increasing sequence. For the converse, we assume that β is stable. Then there exists an $M > 1$ such that

$$\beta_{2n} \leqslant M\beta_n \qquad \forall n \in \mathbb{N}.$$

Since β is increasing we have the following: if $j = 2t \geqslant 2n$,

$$\beta_j = \beta_{2t} \leqslant M\beta_t \leqslant M\beta_{2t-n} = M\beta_{j-n} \leqslant M\beta_{j-n+1}$$

and if $j = 2t + 1 \geqslant 2n$, we have $t + 1 \geqslant n$ and

$$\beta_j = \beta_{2t+1} \leqslant \beta_{2t+2} \leqslant M\beta_{t+1} \leqslant M\beta_{2t+2-n} = M\beta_{j-n+1}.$$

Therefore, we have
$$\beta_j \leqslant M\beta_{j-n+1}$$
for all $j \geqslant 2n$. Now let us assume that $\theta \in \Lambda_\infty(\beta)$ and the condition (3.4) holds. Then, for every $k \in \mathbb{N}$ there exist $m \in \mathbb{N}$, $C_1, C_2 > 0$ such that

$$\begin{aligned}
\|\widehat{T}_\theta e_n\|_k &= \sum_{j=n}^{2n-1} |\theta_{j-n}| e^{k\beta_j} + \sum_{j=2n}^{\infty} |\theta_{j-n}| e^{k\beta_j} \\
&\leqslant \sum_{j=n}^{2n-1} |\theta_{j-n}| e^{k\beta_{2n}} + \sum_{j=2n}^{\infty} |\theta_{j-n}| e^{Mk\beta_{j-n+1}} e^{k\beta_j - Mk\beta_{j-n+1}} \\
&\leqslant e^{Mk\beta_n} \sum_{j=n}^{2n-1} |\theta_{j-n}| + \sum_{j=2n}^{\infty} |\theta_{j-n}| e^{Mk\beta_{j-n+1}} \leqslant e^{Mk\beta_n} \sum_{j=n}^{\infty} |\theta_{j-n}| e^{Mk\beta_{j-n+1}} \\
&\leqslant C_1 a_{n,m} \sum_{j=n}^{\infty} |\theta_{j-n}| e^{Mk\beta_{j-n+1}} \leqslant C_1 \|\theta\|_{Mk} \|e_n\|_m \leqslant C_2 \|e_n\|_m.
\end{aligned}$$

holds for every $n \in \mathbb{N}$. We employed that $\beta_j \leqslant M\beta_{j-n+1}$ for every $j \geqslant 2n$ in the first line. Therefore, $\widehat{T}_\theta e_n \in \Lambda_\infty(\beta)$ for every $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$ there exist $m \in \mathbb{N}$ such that
$$\sup_{n \in \mathbb{N}} \frac{\|\widehat{T}_\theta e_n\|_k}{\|e_n\|_m} < \infty$$
that is, $\widehat{T}_\theta$ is well-defined and continuous by Proposition 2.2. □

We can characterize the compactness of the operators $\widehat{T}_\theta : K(a_{n,k}) \to \Lambda_\infty(\beta)$ as the follows.

Theorem 3.7. *Let $\beta = (\beta_n)_{n\in\mathbb{N}}$ be a stable sequence. $\widehat{T}_\theta : K(a_{n,k}) \to \Lambda_\infty(\beta)$ is compact if and only if $\theta \in \Lambda_\infty(\beta)$ and the following condition holds:*
$$\exists m \in \mathbb{N} \quad \forall k \in \mathbb{N} \quad \exists C > 0 \qquad e^{k\beta_n} \leqslant C a_{n,m} \qquad \forall n \geqslant k. \tag{3.5}$$

Proof. Let assume θ be a sequence and $s_0 = \min\{t : \theta_t \neq 0\}$. If $\widehat{T}_\theta$ is compact, then $\theta \in \Lambda_1(\beta)$ from Proposition 3.1. Proposition 3.2 says that there exists a $m \in N$ such that for all $k \in \mathbb{N}$ there exists a $C > 0$ such that
$$e^{k\beta_{n+s_0}} \leqslant C a_{n,m}$$
for every $n \in \mathbb{N}$. This says that the condition (3.5) is satisfied since β is an increasing sequence.

For the converse, assume that $\theta \in \Lambda_\infty(\beta)$ and (3.5) holds. By the same calculation of the proof of Theorem 3.6, we can write that there exists $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$
$$\sup_{n \geqslant k} \frac{\|\widehat{T}_\theta e_n\|_k}{\|e_n\|_m} < \infty$$
and then
$$\sup_{n \in \mathbb{N}} \frac{\|\widehat{T}_\theta e_n\|_k}{\|e_n\|_m} < \infty.$$
From Lemma 2.1, we can say that $\widehat{T}_\theta : K(a_{n,k}) \to \Lambda_\infty(\beta)$ is compact. □

Proposition 3.8. *Let $\beta = (\beta_n)_{n\in\mathbb{N}}$ be a non-negative increasing stable sequence with $\lim\limits_{n\to\infty} \beta_n = \infty$. Assume that there exists a non-decreasing function $S : \mathbb{N} \to \mathbb{N}$ such that*

$$\forall k \in \mathbb{N} \quad \exists C > 0 \qquad e^{k\beta_n} \leqslant C a_{n,S(k)} \qquad \forall n \in \mathbb{N}. \quad (3.6)$$

Then, the family of operators $\widehat{T}_\theta : K(a_{n,k}) \to \Lambda_\infty(\beta)$ for $\theta \in \Lambda_\infty(\beta)$

$$\widehat{\mathcal{A}}_\infty = \{T_\theta : \theta \in \Lambda_\infty(\beta)\}$$

is $\tilde{S}$-tame, where $\tilde{S}(k) = S(Mk)$ for every $k \in \mathbb{N}$ and here $M = \sup\limits_{n\in\mathbb{N}} \dfrac{\alpha_{2n}}{\alpha_n}$.

Proof. The proof is similar to the proof of Theorem 3.7. Let $\theta \in \Lambda_\infty(\beta)$. In the proof of Theorem 3.6 we showed that for all $k, n \in \mathbb{N}$

$$\|\widehat{T}_\theta e_n\|_k \leqslant e^{Mk\beta_n} \sum_{j=n}^{\infty} |\theta_{j-n}| e^{Mk\beta_{j-n+1}} \leqslant e^{Mk\beta_n} \|\theta\|_{Mk}.$$

Using the condition (3.6), we can write that for all $k \in \mathbb{N}$ there exists a $C > 0$ such that

$$\|\widehat{T}_\theta e_n\|_k \leqslant e^{Mk\beta_n} \|\theta\|_{Mk} \leqslant C a_{n,S(Mk)} = C a_{n,\tilde{S}(k)}.$$

Then we have the following

$$\forall k \quad \exists C > 0 \qquad \|\widehat{T}_\theta e_n\|_k \leqslant C \|e_n\|_{\tilde{S}(k)}$$

for every $\theta \in \Lambda_\infty(\beta)$. This says that the family $\widehat{\mathcal{A}}_\infty$ is $\tilde{S}$-tame. □

4. Operators Defined by an Upper Triangular Toeplitz Matrix

Let $\theta = (\theta_n)_{n\in\mathbb{N}}$ be any sequence. The upper triangular Toeplitz matrix defined by θ is

$$\begin{pmatrix} \theta_0 & \theta_1 & \theta_2 & \theta_3 & \cdots \\ 0 & \theta_0 & \theta_1 & \theta_2 & \cdots \\ 0 & 0 & \theta_0 & \theta_1 & \cdots \\ 0 & 0 & 0 & \theta_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We want to define a linear operator $\check{T}_\theta : K(a_{n,k}) \to K(b_{n,k})$ by taking $\check{T}_\theta e_n$ as the nth column of the above matrix, that is,

$$\check{T}_\theta e_n = (\theta_{n-1}, \theta_{n-2}, \ldots, \theta_1, \theta_0, 0, 0, \ldots) = \sum_{j=1}^{n} \theta_{n-j} e_j.$$

Therefore, for every $x = \sum_{n=1}^{\infty} x_n e_n \in K(a_{n,k})$, the operator $\check{T}_\theta$ is given by

$$\check{T}_\theta x = \sum_{n=1}^{\infty} x_n \check{T}_\theta e_n.$$

In fact, we can not guarantee that the operator $\check{T}_\theta$ is correctly defined between two general Köthe spaces $K(a_{n,k})$ and $K(b_{n,k})$, because we do not know if the series $\sum_{n=1}^{\infty} x_n \check{T}_\theta e_n$ converges in $K(b_{n,k})$ for every $x \in K(a_{n,k})$ in this general case. Below, we will examine the cases where this operator can be correctly defined, and in these cases, we will analyze the continuity and compactness of this operator. Similiar to the previous section, we will assume that the sequences θ satisfy the following condition

$$\forall n \in \mathbb{N} \quad \exists s > n \qquad \theta_s \neq 0.$$

Otherwise, θ produces a finite rank operator, which is obviously continuous and compact.

In [10, Theorem 2.2], you can find the same idea for Proposition 4.1 and a generalized form of Theorem 4.3 for G_∞-spaces.

Proposition 4.1. *Let $K(a_{n,k})$, $K(b_{n,k})$ be Köthe spaces and assume that θ be a sequence and $s_0 = \min\{t : \theta_t \neq 0\}$. If $\check{T}_\theta : K(a_{n,k}) \to K(b_{n,k})$ is continuous, then $\theta \in (K(a_{n,k}))'$ and the following holds:*

$$\forall k \in \mathbb{N} \quad \exists m \in \mathbb{N}, C > 0 \qquad b_{n-s_0,k} \leqslant C a_{n,m} \qquad \forall n > s_0.$$

Proof. Let $\check{T}_\theta : K(a_{n,k}) \to K(b_{n,k})$ be a continuous operator. Then by Lemma 2.1, for all $k \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C_1 > 0$ such that

$$\|\check{T}_\theta e_n\|_k = \sum_{j=1}^{n} |\theta_{n-j}| b_{j,k} \leqslant C_1 a_{n,m} \qquad \forall n \in \mathbb{N}.$$

Then, for all n and $j \leqslant n$

$$|\theta_{n-j}| b_{j,k} \leqslant C_1 a_{n,m}.$$

Since $(b_{n,k})$ is a Köthe matrix, there exists a $k_0 \in \mathbb{N}$ satisfying $b_{1,k_0} \neq 0$. Let us take $C_2 = \frac{C_1}{b_{1,k_0}}$. Then we can write

$$|\theta_{n-1}| \leqslant C_2 a_{n,m}$$

and this says that $\theta \in (K(a_{n,k}))'$. Further, we have that for every $k \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and $C_1 > 0$ so that

$$|\theta_{s_0}| b_{n-s_0} \leqslant C_1 a_{n,m}$$

and

$$b_{n-s_0} \leqslant \frac{C_1}{\theta_{s_0}} a_{n,m}.$$

hold for all $n > s_0$. By choosing $C = \dfrac{C_1}{|\theta_{s_0}|}$, we have

$$b_{n-s_0,k} \leqslant C_3 a_{n,m} \qquad \forall n > s_0.$$

This completes the proof. □

For compact $\check{T}_\theta : K(a_{n,k}) \to K(b_{n,k})$ operators, the relationship between Köthe matrices $(a_{n,k})_{n,k\in\mathbb{N}}$ and $(b_{n,k})_{n,k\in\mathbb{N}}$ is as follows:

Proposition 4.2. *Let $K(a_{n,k})$ and $K(b_{n,k})$ be Köthe spaces such that $K(b_{n,k})$ is Montel and assume that $\theta \in K(b_{n,k})$ and $s_0 = \min\{t : \theta_t \neq 0\}$. If $\check{T}_\theta : K(a_{n,k}) \to K(b_{n,k})$ is a compact operator, then the following holds*

$$\exists m \in \mathbb{N} \quad \forall k \in \mathbb{N} \quad \exists C > 0 \qquad b_{n-s_0,k} \leqslant C a_{n,m} \qquad \forall n > s_0.$$

Proof. The proof is similar to the proof of Proposition 4.1. □

The converse of Proposition 3.1 is true when $K(b_{n,k})$ is a power series space of finite type.

Theorem 4.3. *Let $\Lambda_\infty(\alpha)$ be a nuclear power series space of infinite type, $K(b_{n,k})$ be a Köthe space and assume that θ be a sequence and $s_0 = \min\{t : \theta_t \neq 0\}$. The operator $\check{T}_\theta : \Lambda_\infty(\alpha) \to K(b_{n,k})$ is well-defined and continuous if and only if $\theta \in (\Lambda_\infty(\alpha))'$ and the following condition holds:*

$$\forall k \in \mathbb{N} \quad \exists m \in \mathbb{N}, C > 0 \qquad b_{n-s_0,k} \leqslant C e^{m\alpha_n} \qquad \forall n > s_0. \quad (4.1)$$

Proof. If the operator $\check{T}_\theta$ is continuous, Proposition 4.1 gives us that $\theta \in (\Lambda_\infty(\alpha))'$ and (4.1) holds. Let us assume that $\theta \in (\Lambda_\infty(\alpha))'$ and (4.1) holds. Then there exists some $m_1 \in \mathbb{N}$ and $C_1 > 0$ such that

$$|\theta_{n-1}| \leqslant C_1 e^{m_1\alpha_n} \qquad \forall n \in \mathbb{N}.$$

From the condition (4.1), for every $k \in \mathbb{N}$, there exist $m_2 \in \mathbb{N}$ and $C_2 > 0$ such that

$$\begin{aligned}\|\check{T}_\theta e_n\|_k &= \sum_{j=1}^{n} |\theta_{n-j}| b_{j,k} = \sum_{j=1}^{n-s_0} |\theta_{n-j}| b_{j,k} \leqslant C_1 \sum_{j=1}^{n-s_0} e^{m_1\alpha_{n-j+1}} b_{j,k} \\ &\leqslant C_2 \sum_{j=1}^{n-s_0} e^{m_1\alpha_{n-j+1}} e^{m_2\alpha_{j+s_0}} \leqslant C_2 e^{m_1\alpha_n} \sum_{j=1}^{n-s_0} e^{m_2\alpha_{j+s_0}} \\ &= C_2 e^{m_1\alpha_n} \sum_{j=1+s_0}^{n} e^{m_2\alpha_j}.\end{aligned}$$

On the other hand, $\Lambda_\infty(\alpha)$ is nuclear then $\sum_{n=1}^{\infty} e^{-m_3\alpha_n}$ is convergent for some $m_3 \in \mathbb{N}$ and then for some $D > 0$ we have

$$\sum_{j=1}^{n} e^{m_2\alpha_j} \leqslant D e^{(m_2+m_3)\alpha_n}$$

since

$$\sum_{j=1}^{n} e^{m_2\alpha_j} e^{-(m_2+m_3)\alpha_n} \leqslant \sum_{j=1}^{n} e^{m_2\alpha_j} e^{-(m_2+m_3)\alpha_j} \leqslant \sum_{j=1}^{\infty} e^{-m_3\alpha_j} = D < +\infty.$$

Then we can write

$$\begin{aligned}\|\check{T}_\theta e_n\|_k &\leqslant C_2 e^{m_1\alpha_n} \sum_{j=1+s_0}^{n} e^{m_2\alpha_j} \leqslant C_2 e^{m_1\alpha_n} \sum_{j=1}^{n} e^{m_2\alpha_j} \\ &\leqslant D C_2 e^{m_1\alpha_n} e^{(m_2+m_3)\alpha_n} = D C_2 \|e_n\|_{m_1+m_2+m_3}.\end{aligned}$$

Therefore, $\breve{T}_\theta e_n \in K(b_{n,k})$ for every $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$, there exist $m = m_1 + m_2 + m_3 \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} \frac{\|\breve{T}_\theta e_n\|_k}{\|e_n\|_m} < \infty,$$

that is, $\breve{T}_\theta$ is well defined and continuous by Proposition 2.2. □

We can characterize the compactness of the operators $\breve{T}_\theta : \Lambda_\infty(\beta) \to K(b_{n,k})$ as follows.

Theorem 4.4. *Let $\Lambda_\infty(\alpha)$ be a nuclear power series space of infinite type and $K(b_{n,k})$ be a Montel Köthe space. $\breve{T}_\theta : \Lambda_\infty(\alpha) \to K(b_{n,k})$ is compact if and only if $\theta \in (\Lambda_\infty(\alpha))'$ and the following condition holds:*

$$\exists m \in \mathbb{N} \quad \forall k \in \mathbb{N} \quad \exists C > 0 \qquad b_{n-s_0,k} \leqslant Ce^{m\alpha_n} \qquad \forall n > s_0. \quad (4.2)$$

Proof. If $\breve{T}_\theta$ is compact, $\theta \in (\Lambda_\infty(\alpha))'$ from Proposition 4.1 and Lemma 2.1 says that there exists a $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$

$$\sup_{n} \frac{\|\breve{T}_\theta e_n\|_k}{\|e_n\|_m} < \infty.$$

This gives us that there exists a $C > 0$ such that

$$|\theta_{s_0}| b_{n-s_0,k} \leqslant \|\breve{T}_\theta e_n\| = \sum_{j=1}^{n} |\theta_{n-j}| b_{j,k} \leqslant C\|e_n\|_m = Ce^{m\alpha_n}.$$

This says that the condition (4.2) is satisfied.

For the converse, assume $\theta \in (\Lambda_\infty(\alpha))'$ and the condition (4.2) holds. By the same calculation of the proof of the Theorem 4.3, we can write that there exists a $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$

$$\sup_{n} \frac{\|\breve{T}_\theta e_n\|_k}{\|e_n\|_m} < \infty.$$

Lemma 2.1 gives us that $\breve{T}_\theta : \Lambda_\infty(\alpha) \to K(b_{n,k})$ is compact. □

Proposition 4.5. *Let $\Lambda_\infty(\alpha)$ be a nuclear power series space of infinite type and $K(b_{n,k})$ be a Köthe space. Assume that there exists a non-decreasing function $S : \mathbb{N} \to \mathbb{N}$ such that the following condition holds*

$$\forall k \in \mathbb{N} \quad \exists C > 0 \qquad b_{n,k} \leqslant Ce^{S(k)\alpha_n} \qquad \forall n \in \mathbb{N}. \quad (4.3)$$

Then, the family of the operators $\breve{T}_\theta : \Lambda_\infty(\alpha) \to K(b_{n,k})$ for $\theta \in (\Lambda_\infty(\alpha))'$, that is,

$$\breve{\mathcal{A}}_\infty = \{\breve{T}_\theta : \theta \in (\Lambda_\infty(\alpha))'\}$$

is 2S-tame.

Proof. The proof is similiar to the proof of Theorem 4.3. Let us assume $\theta \in (\Lambda_\infty(\alpha))'$. Then there exists some $m_1 \in \mathbb{N}$ and $C_1 > 0$ such that

$$|\theta_{n-1}| \leqslant C_1 e^{m_1\alpha_n} \qquad \forall n \in \mathbb{N}.$$

Since $\Lambda_\infty(\alpha)$ is nuclear then $\sum_{n=1}^{\infty} e^{-m_2\alpha_n}$ is convergent for some $m_2 \in \mathbb{N}$ and using similiar to steps in the proof of Theorem 4.3 we can show that there exists a $D > 0$ such that

$$\sum_{j=1}^{n} e^{k\alpha_j} \leqslant De^{(k+m_2)\alpha_n}$$

for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ satisfying $S(k) \geqslant m_1 + m_2$, there exists $C_2, C_3 > 0$ such that

$$\begin{aligned}\|\check{T}_\theta e_n\|_k &= \sum_{j=1}^{n} |\theta_{n-j}| b_{j,k} \leqslant C_1 \sum_{j=1}^{n} e^{m_1\alpha_{n-j+1}} b_{j,k} \\ &\leqslant C_2 e^{m_1\alpha_n} \sum_{j=1}^{n} e^{S(k)\alpha_j} \leqslant C_3 e^{m_1\alpha_n} e^{(S(k)+m_2)\alpha_n} \leqslant C_3 e^{2S(k)\alpha_n} = \|e_n\|_{2S(k)}.\end{aligned}$$

Then, we have the following

$$\|\check{T}_\theta e_n\|_k \leqslant C_2 \|e_n\|_{2S(k)}$$

for all k satisfying $S(k) \geqslant m_1 + m_2$, that is, the family $\check{\mathcal{A}}_\infty$ is 2S-tame. □

For the converse of Proposition 4.4, we need a stable sequence α when $K(a_{n,k})$ is power series space of finite type $\Lambda_1(\alpha)$.

Theorem 4.6. *Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a stable sequence. $\check{T}_\theta : \Lambda_1(\alpha) \to K(b_{n,k})$ is well-defined and continuous if and only if $\theta \in (\Lambda_1(\alpha))'$ and the following condition holds*

$$\forall k \in \mathbb{N} \quad \exists m \in \mathbb{N}, C > 0 \qquad b_{n,k} \leqslant Ce^{-\frac{1}{m}\alpha_n} \qquad \forall n \in \mathbb{N}. \tag{4.4}$$

Proof. Let assume θ be a sequence and $s_0 = \min\{t : \theta_t \neq 0\}$. If the operator $\check{T}_\theta$ is continuous, Proposition 4.1 gives us that $\theta \in (\Lambda_1(\alpha))'$ and for every $k \in \mathbb{N}$, there exist $m \in \mathbb{N}$, $C > 0$ such that

$$b_{n,k} \leqslant Ce^{-\frac{1}{m}\alpha_{n+s_0}} \leqslant Ce^{-\frac{1}{m}\alpha_n}$$

for every $n \in \mathbb{N}$. This says that (4.4) holds since β is an increasing sequence. For the converse, we assume that assume that α is stable, $\theta \in (\Lambda_1(\alpha))'$ and (4.4) holds. Then there exist some $m_1 \in \mathbb{N}$ and $C_1 > 0$ such that

$$|\theta_{n-1}| \leqslant C_1 e^{-\frac{1}{m_1}\alpha_n} \qquad \forall n \in \mathbb{N}.$$

By stability of α, we will show that there exists $M > 0$ such that for every $n \in \mathbb{N}$ and $j \leqslant n$ it holds that

$$\alpha_n \leqslant M(\alpha_{n-j+1} + \alpha_j). \tag{4.5}$$

Since α is stable, there exists a $M > 0$ such that $\alpha_{2t} \leqslant M\alpha_t$ for every $t \in \mathbb{N}$. Assume that $n = 2t$ or $n = 2t + 1$ and $1 \leqslant j \leqslant n$

$$\alpha_n \leqslant \alpha_{2t+2} \leqslant M\alpha_{t+1} \leqslant M(\alpha_{n-j+1} + \alpha_j)$$

since in this case $t+1 \leqslant j$ or $t+1 \leqslant n-j+1$ and then we have $\alpha_{n-j+1}+\alpha_j \geqslant \alpha_{t+1}$. Therefore the inequality (4.5) is satisfied. By using the condition (4.4), we can write that for every $k \in \mathbb{N}$, there exist $m_2 \in \mathbb{N}$ and $C_2 > 0$ such that

$$b_{n,k} \leqslant C_2 e^{-\frac{1}{m_2}\alpha_{n+s_0}} \leqslant C_2 e^{-\frac{1}{m_2}\alpha_n}$$

for every $n \in \mathbb{N}$. Let us say $m_3 = \min(m_1, m_2)$. Then we have

$$\begin{aligned}\|\check{T}_\theta e_n\|_k = \sum_{j=1}^{n} |\theta_{n-j}| b_{j,k} &\leqslant C_1 C_2 \sum_{j=1}^{n} e^{-\frac{1}{m_1}\alpha_{n-j+1}} e^{-\frac{1}{m_2}\alpha_j} \\ &\leqslant C_1 C_2 \sum_{j=1}^{n} e^{-\frac{1}{m_3}(\alpha_{n-j+1}+\alpha_j)} \\ &\leqslant C_1 C_2 \sum_{j=1}^{n} e^{-\frac{1}{Mm_3}\alpha_n} \leqslant C_1 C_2 n e^{-\frac{1}{m_3 M}\alpha_n}\end{aligned}$$

Now we choose a $m_4 \in \mathbb{N}$ so that $m_4 > m_3 M$, then

$$-\frac{1}{m_3 M} + \frac{1}{m_4} < 0 \qquad \text{and} \qquad \lim_{n\to\infty} n e^{(-\frac{1}{m_3 M}+\frac{1}{m_4})\alpha_n} = 0.$$

Then there exists a $D > 0$ such that for all $n \in \mathbb{N}$

$$n e^{-\frac{1}{m_3 M}\alpha_n} \leqslant D e^{-\frac{1}{m_4}\alpha_n}$$

and for some $C_3 > 0$ and for all $n \in \mathbb{N}$

$$\|\check{T}_\theta e_n\|_k \leqslant C_1 C_2 n e^{-\frac{1}{m_3 M}\alpha_n} \leqslant C_3 e^{-\frac{1}{m_4}\alpha_n} \leqslant C_3 \|e_n\|_{m_4}.$$

Therefore, for all $k \in \mathbb{N}$, there exist $m \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} \frac{\|\check{T}_\theta e_n\|_k}{\|e_n\|_m} < \infty,$$

that is, $\check{T}_\theta : \Lambda_1(\alpha) \to K(b_{n,k})$ is well-defined and continuous by Proposition 2.2. □

The following theorem characterize the compactness of the operators $\check{T}_\theta : \Lambda_1(\alpha) \to K(b_{n,k})$.

Theorem 4.7. *Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a stable sequence and $K(b_{n,k})$ be a Montel Köthe space. $\check{T}_\theta : \Lambda_1(\alpha) \to K(b_{n,k})$ is compact if and only if $\theta \in (\Lambda_1(\alpha))'$ and the following condition holds:*

$$\exists m \in \mathbb{N} \quad \forall k \in \mathbb{N} \quad \exists C > 0 \qquad b_{n,k} \leqslant C e^{-\frac{1}{m}\alpha_n} \qquad \forall n \geqslant k. \quad (4.6)$$

Proof. Let assume θ be a sequence and $s_0 = \min\{t : \theta_t \neq 0\}$. If the operator $\widehat{T}_\theta$ is continuous, then $\theta \in (\Lambda_1(\alpha))'$ from Proposition 4.1 and Proposition 4.2 says that for every $k \in \mathbb{N}$, there exist $m \in \mathbb{N}$, $C > 0$ such that

$$b_{n,k} \leqslant C e^{-\frac{1}{m}\alpha_{n+s_0}} \leqslant C e^{-\frac{1}{m}\alpha_n}$$

for every $n \in \mathbb{N}$. This says that (4.6) holds since α is an increasing sequence.

For the converse, assume that $\theta \in (\Lambda_1(\alpha))'$ and the condition (4.6) holds. By the same calculation of the proof of the Theorem 4.6, we can write that there exists $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$

$$\sup_n \frac{\|\check{T}_\theta e_n\|_k}{\|e_n\|_m} < \infty,$$

that is, $\check{T}_\theta : \Lambda_1(\alpha) \to K(a_{n,k})$ is compact from Lemma 2.1. □

Proposition 4.8. *Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a stable sequence and $M = \sup_{n\in\mathbb{N}} \frac{\alpha_{2n}}{\alpha_n}$. Assume that there exists a non-decreasing function $S : \mathbb{N} \to \mathbb{N}$ such that the following condition hold:*

$$\forall k \in \mathbb{N} \quad \exists C > 0 \qquad b_{n,k} \leqslant C e^{-\frac{1}{S(k)}\alpha_n} \qquad \forall n \in \mathbb{N}. \quad (4.7)$$

Then, the family of the operators $\check{T}_\theta : \Lambda_1(\alpha) \to K(b_{n,k})$ for $\theta \in (\Lambda_1(\beta))'$, that is,

$$\check{\mathcal{A}}_1 = \{\check{T}_\theta : \theta \in (\Lambda_1(\alpha))'\}$$

is $\tilde{S}$-tame where $\tilde{S}(k) = (M+1)S(k)$ for every $k \in \mathbb{N}$.

Proof. The proof is similar to the proof of the Theorem 4.6. Since $\theta \in (\Lambda_1(\alpha))'$, there exists some $m_1 \in \mathbb{N}$ and $C_1 > 0$ such that

$$|\theta_{n-1}| \leqslant C_1 e^{-\frac{1}{m_1}\alpha_n} \qquad \forall n \in \mathbb{N}.$$

By using (4.7), we can write that for every $k \in \mathbb{N}$ satisfying $S(k) \geqslant m_1$, there exists a $C_2, C_3, C_4 > 0$ such that

$$\begin{aligned}
\|\check{T}_\theta e_n\|_k &= \sum_{j=1}^n |\theta_{n-j}| b_{j,k} \leqslant C_1 \sum_{j=1}^n e^{-\frac{1}{m_1}\alpha_{n-j+1}} b_{j,k} \leqslant C_2 \sum_{j=1}^n e^{-\frac{1}{m_1}\alpha_{n-j+1}} e^{-\frac{1}{S(k)}\alpha_j} \\
&\leqslant C_2 \sum_{j=1}^n e^{-\frac{1}{S(k)}(\alpha_{n-j+1}+\alpha_n)} \leqslant C_3 \sum_{j=1}^n e^{-\frac{1}{MS(k)}\alpha_n} = C_3 n e^{-\frac{1}{MS(k)}\alpha_n} \\
&\leqslant C_4 e^{-\frac{1}{(M+1)S(k)}\alpha_n} \leqslant C_4 \|e_n\|_{(M+1)S(k)}
\end{aligned}$$

Then for all $k \in \mathbb{N}$ satisfying $S(k) \geqslant m_1$, then, we have the following

$$\|\check{T}_\theta e_n\|_k \leqslant C_3 \|e_n\|_{(M+1)S(k)}$$

that is the family $\check{\mathcal{A}}_1$ is $\tilde{S}$-tame, where $\tilde{S}(k) = (M+1)S(k)$ for all $k \in \mathbb{N}$. □

5. Toeplitz Operators Defined between Power Series Spaces

When given a Toeplitz matrix

$$\begin{pmatrix} \theta_0 & \theta_{-1} & \theta_{-2} & \cdots \\ \theta_1 & \theta_0 & \theta_{-1} & \cdots \\ \theta_2 & \theta_1 & \theta_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

one can express this matrix as a sum of a lower and an upper Toeplitz matrix in the following way:

$$\begin{pmatrix}\theta_0 & \theta_{-1} & \theta_{-2} & \cdots \\ \theta_1 & \theta_0 & \theta_{-1} & \cdots \\ \theta_2 & \theta_1 & \theta_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots\end{pmatrix} = \begin{pmatrix}\theta_0' & 0 & 0 & \cdots \\ \theta_1 & \theta_0' & 0 & \cdots \\ \theta_2 & \theta_1 & \theta_0' & \cdots \\ \vdots & \vdots & \vdots & \ddots\end{pmatrix} + \begin{pmatrix}\theta_0'' & \theta_{-1} & \theta_{-2} & \cdots \\ 0 & \theta_0'' & \theta_{-1} & \cdots \\ 0 & 0 & \theta_0'' & \cdots \\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}$$

Here $\theta_0 = \theta_0' + \theta_0''$ and we can choose θ_0' and θ_0'' to be non-zero. Let define the sequences

$$\widehat{\theta} = (\theta_0', \theta_1, \theta_2, \theta_3, \dots)$$

and

$$\widecheck{\theta} = (\theta_0'', \theta_{-1}, \theta_{-2}, \theta_{-3}, \dots).$$

Therefore every Toeplitz matrix can be associated with two sequences. Denoting the operator defined by the Toeplitz matrix as T_θ, one can write this operator as a sum of two operators defined by a lower and an upper Toeplitz matrix, that is,

$$T_\theta = \widehat{T}_{\widehat{\theta}} + \widecheck{T}_{\widecheck{\theta}}.$$

In this section, we collect some results regarding the continuity and compactness of a Toeplitz operator by utilizing the theorems proven in the previous sections.

Theorem 5.1. *Let α be a stable sequence. A Toeplitz operator $T_\theta : \Lambda_1(\alpha) \to \Lambda_1(\beta)$ is well-defined and continuous if $\theta \in \Lambda_1(\beta) \oplus (\Lambda_1(\alpha))'$ and the following condition holds:*

$$\forall k \in \mathbb{N} \quad \exists m \in \mathbb{N}, C > 0 \qquad e^{-\frac{1}{k}\beta_n} \leqslant C e^{-\frac{1}{m}\alpha_n} \qquad \forall n \in \mathbb{N}. \quad (5.1)$$

Proof. Let assume that Toeplitz matrix associated with T_θ is given by sequnces $\widehat{\theta}$ and $\widecheck{\theta}$. We can choose θ_0' and θ_0'' to be non-zero. Then Toeplitz operator T_θ is defined as

$$T_\theta = \widehat{T}_{\widehat{\theta}} + \widecheck{T}_{\widecheck{\theta}}.$$

From Theorem (3.3), $\widehat{T}_{\widehat{\theta}}$ is well-defined and continous if and only if $\widehat{\theta} \in \Lambda_1(\beta)$ and the condition (5.1) is satisfied. From Theorem 4.6, $\widecheck{T}_{\widecheck{\theta}}$ is well-defined and continous if and only if $\widecheck{\theta} \in (\Lambda_1(\alpha))'$ and the condition (5.1) is satisfied. Therefore, these give us that T_θ is well-defined and continuous if $\theta = \widehat{\theta} \oplus \widecheck{\theta} \in \Lambda_1(\beta) \oplus (\Lambda_1(\alpha))'$ and the condition (5.1) is satisfied. □

Theorem 5.2. *Let α be a stable sequence. A Toeplitz operator $T_\theta : \Lambda_1(\alpha) \to \Lambda_1(\beta)$ is compact if a $\theta \in \Lambda_1(\beta) \oplus (\Lambda_1(\alpha))'$ and the following condition holds:*

$$\exists m \in \mathbb{N} \quad \forall k \in \mathbb{N} \quad \exists C > 0 \qquad e^{-\frac{1}{k}\beta_n} \leqslant C e^{-\frac{1}{m}\alpha_n} \qquad \forall n \geqslant k. \quad (5.2)$$

Proof. The proof is similar to the proof of Theorem 5.1 and follows from Theorem 3.4 and Theorem 4.7. □

Remark 5.1. *The condition in Theorem 5.2 is satisfied for the sequences $\alpha = (n)_{n \in \mathbb{N}}$ and $\beta = (n^2)_{n \in \mathbb{N}}$.*

Theorem 5.3. *Let β be a stable sequence and $\Lambda_\infty(\alpha)$ be a nuclear power series space of infinite type. A Toeplitz operator $T_\theta : \Lambda_\infty(\alpha) \to \Lambda_\infty(\beta)$ is well-defined and continuous if $\theta \in \Lambda_\infty(\beta) \oplus (\Lambda_\infty(\alpha))'$ and the following condition holds:*

$$\forall k \in \mathbb{N} \quad \exists m \in \mathbb{N}, C > 0 \qquad e^{k\beta_n} \leqslant C e^{m\alpha_n} \qquad \forall n \geqslant k. \tag{5.3}$$

Proof. This follows from Theorem 3.6 and Theorem 4.3. □

Theorem 5.4. *Let β be a stable sequence and $\Lambda_\infty(\alpha)$ be a nuclear power series space of infinite type. A Toeplitz operator $T_\theta : \Lambda_\infty(\alpha) \to \Lambda_\infty(\beta)$ is compact if $\theta \in \Lambda_\infty(\beta) \oplus (\Lambda_\infty(\alpha))'$ and the following condition holds:*

$$\exists m \in \mathbb{N} \quad \forall k \in \mathbb{N} \quad \exists C > 0 \qquad e^{k\beta_n} \leqslant C e^{m\alpha_n} \qquad \forall n \geqslant k. \tag{5.4}$$

Proof. This follows from Theorem 3.7 and Theorem 4.4. □

Remark 5.2. *The condition in Theorem 5.4 is satisfied for the sequences $\alpha = (n^2)_{n\in\mathbb{N}}$ and $\beta = (n)_{n\in\mathbb{N}}$.*

Theorem 5.5. *Let $\Lambda_\infty(\alpha)$ be a nuclear power series space of infinite type and $\Lambda_1(\beta)$ be a nuclear power series spaces of finite type. $T_\theta : \Lambda_\infty(\alpha) \to \Lambda_1(\beta)$ is well-defined, continuous and compact if $\theta \in \Lambda_1(\beta) \oplus (\Lambda_\infty(\beta))'$.*

Proof. Continuity follows from Theorem 3.3 and Theorem 4.3. Compactness follows from Theorem 3.4 and Theorem 4.4. □

Remark 5.3. *By considering the conditions in Proposition 3.1 or Proposition 4.1, the Toeplitz operator T_θ is not well defined from $\Lambda_1(\alpha)$ to $\Lambda_\infty(\beta)$.*

Acknowledgments

The results in this paper were obtained while the author visited at University of Toledo. I would like to thank TUBITAK for their support and Prof. Dr. Sönmez Şahutoğlu for sharing Jasiczak's paper with me, which enabled me to create this work.

References

[1] A. Aytuna, *Tameness in Fréchet spaces of analytic functions*, Studia Math., **232**, 2016, 243-266.

[2] L. Crone and W. Robinson, *Diagonal maps and diameters in Köthe spaces* , Israel J. Math., **20**, 1975, 13-22.

[3] P. Domański and M. Jasiczak, *Toeplitz operators on the space of real analytic functions: the Fredholm property.* Banach J. Math. Anal. 12, 2018, **1**, 31-67.

[4] E. Dubinsky and D. Vogt, *Complemented subspaces in tame power series spaces*, Studia Mathematica, 1(93), 1989, 71–85.

[5] R. S. Hamilton. *The inverse function theorem of Nash and Moser.* American Mathematical Society, 7(1), 1982.

[6] M. Jasiczak, *Toeplitz operators on the space of all entire functions.* N. Y. J. Math. 26, 2020, 756–789.

[7] K. Piszczek. *On tame pairs of Fréchet spaces.* Mathematische Nachrichten, 282(2), 2009, 270–287.

[8] R. A. Martínez-Avendaño and P. Rosenthal, *An Introduction to Operators on the Hardy-Hilbert Space*, Springer New York, NY, 2010.

[9] R. Meise and D. Vogt, *Introduction to functional analysis*, Clarendon Press, Oxford, 1997.

[10] E. Uyanık and M. H. Yurdakul, *A note on triangular operators on smooth sequence spaces*, Operators and Matrices, **13(2)**, 2019, 343-347.

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