Proc. Indian Acad. Sci. (Math. Sci.) Vol. 119, No. 2, April 2009, pp. 149–158.

On a class of semicommutative modules

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MS received 13 November 2007; revised 24 September 2008

Abstract. Let R be a ring with identity, M a right R-module and $S = \mathrm{End}_R(M)$. In this note, we introduce S-semicommutative, S-Baer, S-q.-Baer and S-p.q.-Baer modules. We study the relations between these classes of modules. Also we prove if M is an S-semicommutative module, then M is an S-p.q.-Baer module if and only if $M[x]$ is an $S[x]$-p.q.-Baer module, M is an S-Baer module if and only if $M[x]$ is an $S[x]$-Baer module, M is an S-q.-Baer module if and only if $M[x]$ is an $S[x]$-q.-Baer module.

Keywords. Baer modules; principally quasi-Baer modules; quasi-Baer modules; semicommutative modules.

1. Introduction

Throughout this paper R will denote an associative ring with identity, Mod-R will be the category of unitary right R-modules. For a module M, $S = \mathrm{End}_R(M)$ will denote the ring of right R-module endomorphisms of M. Then M is a left S-module, right R-module and S-R-bimodule. In this work, for any rings S and R and any S-R-bimodule M, $r_R(.)$ and $l_M(.)$ will denote the right annihilator of a subset of M with elements from R and the left annihilator of a subset of R with elements from M, respectively. Similarly, $l_S(.)$ and $r_M(.)$ will be the left annihilator of a subset of M with elements from S and the right annihilator of a subset of S with elements from M, respectively. In [10], Rizvi and Roman called M a *Baer module* if the right annihilator in M of any left ideal of S is generated by an idempotent of S, i.e., for any left ideal I of S, $r_M(I) = eM$ for some $e^2 = e \in S$ (or equivalently, for all R-submodules N of M, $l_S(N) = Se$ with $e^2 = e \in S$). M is said to be a *quasi-Baer module* if the right annihilator in M of any ideal of S is generated by an idempotent of S (or equivalently, for all fully invariant R-submodules N of M, $l_S(N) = Se$ with $e^2 = e \in S$). To avoid confusion with definitions in [6], we will call Baer modules S-*Baer* modules and quasi-Baer modules S-*quasi-Baer* modules. Among other results they have proved that any direct summand of an S-Baer (resp. S-quasi-Baer) module M is again an S-Baer (resp. S-quasi-Baer) module, and the endomorphism ring $S = \mathrm{End}_R(M)$ of an S-Baer (resp. S-quasi-Baer) module M is an S-Baer (resp. S-quasi-Baer) ring (see Theorem 4.1 in [10]). They gave several results for a direct sum of S-Baer (resp. S-quasi-Baer) modules to be an S-Baer (resp. S-quasi-Baer) module.

Let M be an R-module. Recall that M is called a *semicommutative module* if for any $a \in R$ and $m \in M$, $ma = 0$ implies $mRa = 0$ and R is called a semicommutative ring if R_R is a semicommutative module. In this work we will call M *S-semicommutative* if for any $f \in S$ and $m \in M$, $f(m) = 0$ implies $fg(m) = 0$ for every $g \in S$. Then a ring R is a semicommutative ring if and only if R_R is an S-semicommutative module where $S = \mathrm{End}_R(R_R) \cong R$. Note that any submodule N of an S-semicomutative module M is S-semicomutative. M is *S-principally quasi-Baer* (or S-p.q.-*Baer* for short) if for any $m \in M$, $l_S(m) = Se$ (which is equal to $l_S(mR)$) for some $e^2 = e \in S$. A ring is called an *abelian* ring if its idempotents are central. And also note that if M is an S-semicommutative module, then for all $\alpha \in S$, $\mathrm{Ker}(\alpha)$ is a fully invariant submodule of M. In particular every direct summand of M is a fully invariant submodule of M and so M satisfies summand intersection property, that is, intersection of two direct summand of M is again direct summand.

2. Preliminaries

In this section we study some elementary properties of S-semicommutative modules. We start with

Lemma 2.1. *Let M be an S-semicommutative module. Then S is a semicommutative*, *hence an abelian ring.*

Proof. Let $f, g \in S$ and assume $fg = 0$. Then $fg(m) = 0$ for all $m \in M$. By hypothesis $fhg(m) = 0$ for all $m \in M$ and $h \in S$. Hence $fhg = 0$ for all $h \in S$ and so $fSg = 0$. Let $e, f \in S$ with $e^2 = e$. Then $(e(1-e))M = 0$. By hypothesis $(ef(1-e))M = 0$. Hence $ef(1-e) = 0$ for all $f \in S$. Similarly $(1-e)fe = 0$ for all $f \in S$. Thus $ef = fe$ for all $f \in S$. □

We do not know whether or not the converse of Lemma 2.1 is true in general. Now we investigate at least when the converse of Lemma 2.1 is possible.

Lemma 2.2. *Let R be a ring and eRe be a semicommutative subring where $e^2 = e \in R$. If $ere = 0$ implies $er = 0$, then eR is an S-semicommutative module where $r \in R$, $S = \mathrm{End}_R(eR)$.*

Proof. Let $f(er) = 0$ where $f \in S$. Then for all $g \in S$, $fg(er) = er_1er_2er$ where $f(e) = er_1$ and $g(e) = er_2$. Since $S = \mathrm{End}_R(eR) \cong eRe$, eRe is a semicommutative ring. Also since $f(er) = 0$, $er_1er = 0$. Thus $er_1ere = 0$ and $er_1er_2ere = 0$ for all $er_2e \in eRe$. By the hypothesis of lemma, $er_1er_2er = 0$. Therefore $fg(er) = er_1er_2er = 0$. □

Let R be a ring without identity. If $r_1Rr_2 = 0$ whenever $r_1r_2 = 0$, then it will be called that R has the semicommutative property.

Lemma 2.3. *Let $e^2 = e \in R$ and $S = \mathrm{End}_R(eR)$. Then*

(1) *If eR is a semicommutative module* (*and so eRe is a semicommutative ring*), *then eR is an S-semicommutative module.*
(2) *Let R be a ring and Re be a semicommutative module where $e^2 = e \in R$ but eR has not the semicommutative property. Then eR is not an S-semicommutative module where $S = \mathrm{End}_R(eR) \cong eRe$ but S is a semicommutative ring.*

Proof.

(1) Let $f \in S$ and $f(er_1) = 0$ where $r \in R$. Then $f(e)er = 0$. Let $g \in S$. By $S = \mathrm{End}_R(eR) \cong eRe$, $g(e) = ese$ and $f(e) = ete$ and so $f(g(er)) = er_3eser$ and $f(er) = eter = 0$. Since eR is semicommutative, $eteser = 0$ and therefore $f(g(er)) = 0$. Then eR is an S-semicommutative module.

(2) Assume that eR is an S-semicommutative module where $S = \mathrm{End}_R(eR)$. Take any elements $r_1, r_2 \in R$ such that $er_1er_2 = 0$. Since $S = \mathrm{End}_R(eR) \cong eRe$, then for fixed $r \in R$, note that $f\colon eR \to eR$, $f(e) = ere$, $f(es) = eres$, $s \in S$ is an R-homomorphism. So, if we take $f(e) = er_1e$, then $f(er_2) = er_1er_2 = 0$. Since eR is an S-semicommutative $module$, for all $g(e) = er_3e \in S$ we obtain that $fg(er_2) = 0$ and so $fg(er_2) = er_1er_3er_2 = 0$. Thus we obtain that if $er_1r_2 = 0$, then for all $er_3 \in eR$, $er_1er_3er_2 = 0$. So eR has the semicommutative property. This is a contradiction. Therefore eR is not an S-semicommutative module. But since Re is a semicommutative module and $S = \mathrm{End}_R(eR) \cong eRe$, eRe is a semicommutative subring of R, S is a semicommutative ring. □

We investigate in Lemma 2.4 the conditions under which the semicommutativity of S implies S-semicommutativity of M.

Lemma 2.4. *Let M be a module with endomorphism ring S. Then the following are satisfied*:

(1) *Assume that S is a semicommutative ring, and for every $m \in M$, there exists $g \in S$ such that $g(M) = mR$, then M is an S-semicommutative module.*

(2) *If M is an S-p.p. module and S is a semicommutative ring, then M is an S-semicommutative module.*

(3) *If M is an indecomposable S-Baer module, then M is an S-semicommutative module and so S is semicommutative.*

(4) *Let M be an S-semicommutative module. Assume that for every submodule N of M there exist $e^2 = e \in S$, and $\alpha \in S$ such that $N \subseteq eM$ and $\alpha(N) = eM$. Then M is a Baer module.*

(5) *If M is an S-semicommutative module and every fully invariant submodule is a direct summand of M, then M is an S-Baer module.*

Proof.

(1) Let $f(m) = 0$ where $S = \mathrm{End}_R(M)$. Then by theorem there exists $g \in S$ such that $g(M) = mR$ and so $f(g(mR)) = f(g(M)) = 0$, that is $fg = 0$. Since S is a semicommutative ring for all $h \in S$, $fhg = 0$ and therefore $fh(m) = 0$. Thus M is an S-semicommutative module.

(2) Let $\varphi(m) = 0$ where $\varphi \in S$ and $m \in M$. Since M is an S-p.p. module, there exists $e^2 = e \in S$ such that $l_S(mR) = Se$. Since $\varphi(m) = 0$, $\varphi \in l_S(mR) = Se$ and then $\varphi\beta \in Se\beta$ for all $\beta \in S$. Since S is semicommutative, $e\beta = \beta e$ for all $\beta \in S$ and so $\varphi\beta \in S\beta e \subseteq Se = l_S(mR)$. This implies that $\varphi\beta(m) = 0$.

(3) Let $\varphi(m) = 0$ where $\varphi \in S$ and $m \in M$. Then $\varphi \in l_S(m) = Se$ for some $e^2 = e$. Hence $M = eM \oplus (1 - e)M$ and so $e = 0$ or $e = 1$. It follows that $\varphi = 0$ or $m = 0$.

(4) Let N be a submodule of M. Then there exists an idempotent homomorphism $e \in S$ and $\alpha \in S$ such that $N \subseteq eM$ and $\alpha(N) = eM$. We prove that $l_S(N) = S(1-e)$. It is trivial that $S(1-e) \leq l_S(N)$ since $N \subseteq eM$. Let $\beta \in l_S(N)$. By hypothesis $\beta(N) = 0$ implies $\beta\alpha(N) = 0$. Then $\beta\alpha(N) = \beta eM = 0$, and so $\beta e = 0$. Hence $\beta = \beta(1-e) \in S(1-e)$. So $l_S(N) \leq S(1-e)$. This completes the proof.

(5) Since M is an S-semicommutative module, if $f(n) = 0$ where $f \in S$, then for all $g \in S$, $f(g(n)) = 0$. This implies that for all $f \in S$, $\mathrm{Ker}(f)$ is a fully invariant submodule of M. Let I be an ideal of S. Since $r_M(I) = \cap_{\alpha \in I}\mathrm{Ker}(\alpha)$ and all the $\mathrm{Ker}(\alpha)$ are fully invariant submodules of M, $r_M(I)$ is a fully invariant submodule of M. So it is a direct summand of M and therefore M is an S-Baer module. □

Lemma 2.5. *Let M_R be a cyclic module. Assume that either M is a semicommutative R-module or R is a commutative ring. Then M is an S-semicommutative module if and only if S is a semicommutative ring.*

Proof. Let $M = xR$. Assume that S is a semicommutative ring , and let $f \in S, m \in M$ with $f(m) = 0$. Then $m = xt$ for some $t \in R$. Define $g(xs) = ms$ where $xs \in M$. We prove M is S-semicommutative. For if $x \in M$ and $r \in R$ with $xr = 0$, then, by hypothesis, $xtr = 0$ so $g(xr) = mr = xtr = 0$. Hence g becomes a well-defined endomorphism of M in the cases where M is a semicommutative R-module or R is a commutative ring. But then $0 = f(m) = fg(x)$. Hence $fg = 0$. By assumption $fhg = 0$ for every $h \in S$. So $0 = fhg(x) = fh(m) = 0$. Thus M is an S-semicommutative module. The rest is clear from Lemma 2.1. □

The following two examples shows that it is not necessary that if M is a semicommutative R-module, then M is an S-semicommutative R-module and if M is an S-semicommutative R-module, then M is a semicommutative R-module, respectively.

Example A. There exists a semicommutative R-module M such that it is not S-semicommutative.

Proof. Let F be a field and $R = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$ where F is a field and $M = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ and $S = \mathrm{End}_R(M)$. Then M is a right R-module by usual matrix operations. Let $f, g \in S$ be defined by

$$f\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \; g\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \in M.$$

Then $f\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $fg\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. That is, M is not S-semicommutative. Since R is commutative, M is a semicommutative R-module.

Example B. There exists a module M with $S = \mathrm{End}_R(M)$ such that M is S-semicommutative but not semicommutative.

Proof. Let $\mathbb{Z}$ denote the ring of integers, $M = \mathbb{Z} \times \mathbb{Z}$, $R = \mathrm{End}_{\mathbb{Z}}(\mathbb{Z} \times \mathbb{Z})$ and $S = \mathrm{End}_R(\mathbb{Z} \times \mathbb{Z})$. Then M is an S-semicommutative module. But M_R is not a semicommutative R-module. For if, let f and $g \in R$ be defined by $(a, b)f = (a, 0)$ and $(a, b)g = (b, 0)$ where $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. Then $(0, 1)f = (0, 0)$ but $(0, 1)gf \neq (0, 0)$. Therefore M is not a semicommutative R-module. □

Lemma 2.6. *Let M be an S-semicommutative module. Then the following are satisfied*:

(1) *If M is a quasi-injective module, then every submodule N of M is an S'-semicommutative module where $S' = \mathrm{End}_R(N)$.*
(2) *If $M = R$, then for all $a \in R$, aR is an S'-semicommutative module where $S' = \mathrm{End}_R(aR)$.*

Proof.

(1) Let $f \in S'$ and $f(n) = 0$, $n \in N$. Since f is an endomorphism from N to M and M is a quasi-injective module, f is extended to the function $\bar{f} \in S$ such that $\bar{f}(n') = f(n')$ for all $n' \in N$ and also for all $g \in S'$, there exists a function $\bar{g} \in S$ such that $\bar{g}(n') = g(n')$ for all $n' \in N$. Since M is an S-semicommutative module, $\bar{f}\bar{g}(n) = 0$ for all $g \in S'$. This implies that $\bar{f}\bar{g}(n) = fg(n) = 0$ for all $g \in S'$.
(2) Since R is S-semicommutative, S is semicommutative. Let $f(ar) = 0$ where $f \in \mathrm{End}_R(aR)$ and $ar \in aR$. Then for all $g \in \mathrm{End}_R(aR)$, $fg(ar) = ar_1r_2r$ where $f(a) = ar_1$ and $g(a) = ar_2$. Since $f(ar) = 0$, $ar_1r = 0$ and $S \cong R$ is semicommutative, we get $ar_1r_2r = 0$. This completes the proof. □

COROLLARY 2.7

Every direct summand $M_R^{'}$ of M_R is S'-semicommutative, where $S' = \mathrm{End}_R(M')$.

Proof. From the proof of Lemma 2.6(1) we conclude that direct summand M' of M is also S'-semicommutative with respect to its endomorphism ring $S' = \mathrm{End}(M')$. □

The following example shows that if M is an S-semicommutative module, then any submodule N of M may not be a T-semicommutative module where $S = \mathrm{End}_R(M)$ and $T = \mathrm{End}_R(N)$.

Example C. There exists a module M with a submodule N, $S = \mathrm{End}_R(M)$ and $T = \mathrm{End}_R(N)$ such that M is S-semicommutative, but N is not T-semicommutative.

Proof. Let F be any field, $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} | a, b, c, d \in F \right\}$, $M = R_R$ and $N = \begin{pmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$. Then $\mathrm{End}_R(M) \cong R$ and M is R-semicommutative by [1]. Let $f \in T$ be defined by

$$f \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \in N.$$

Then $f \in T$. Let $N = N_1 \oplus N_2$ where $N_1 = \begin{pmatrix} 0 & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $N_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$ and $e \in T$ be the projection of N onto N_2, i.e., $e(n_1 + n_2) = n_2$ where $n_1 + \in N_1$ and $n_2 \in N_2$. Then $f \in T$ and $e\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$. But $ef\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$. Hence N is not T-semicommutative.

□

Lemma 2.8. *Let $M = M_1 \oplus M_2$, M_1 be S_1-semicommutative and M_2 be S_2-semicommutative, where $S_1 = \mathrm{End}_R(M_1)$ and $S_2 = \mathrm{End}_R(M_2)$. If* $\mathrm{Hom}(M_i, M_j) = 0$ *for $1 \leq i \neq j \leq 2$, then M is S-semicommutative, where $S = \mathrm{End}_R(M)$.*

Proof. Let $fm = 0$ and $S = S_1 \oplus S_2$. Then $f = f_1 + f_2$ and $m = m_1 + m_2$ and $fm = f_1m_1 + f_2m_2$. So $f_1m_1 = 0$, $f_2m_2 = 0$. By hypothesis $f_1S_1m_1 = 0$, $f_2S_2m_2 = 0$. Hence $fSm = 0$. □

COROLLARY 2.9

Let e be an idempotent in a ring R. Then R is a semicommutative ring if and only if e is a central idempotent, eR and $(1-e)R$ are semicommutative rings.

Lemma 2.10. *Let $M = M_1 \oplus M_2$. If* $\mathrm{Hom}(M_2, M_1) = 0$ *or* $\mathrm{Hom}(M_1, M_2) = 0$, *then* $S = \mathrm{End}_R(M)$ *is not semicommutative.*

Proof. Let $0 \neq f \in \mathrm{Hom}(M_2, M_1)$ with $f(m_2) \neq 0$ where $m_2 \in M_2$. Then $(\pi_1 f \pi_2)(m_2) = \pi_1(f(m_2)) = f(m_2)$. Hence $\pi_1 f \pi_2 \neq 0$. This implies that S is not semicommutative since $\pi_1\pi_2 = 0$. □

COROLLARY 2.11

Let $M = M_1 \oplus M_2$. If $S = \mathrm{End}_R(M)$ is a semicommutative ring, then $\mathrm{Hom}(M_i, M_j) = 0$ *where* $1 \leq i \neq j \leq 2$.

Lemma 2.12. *Let M be a duo module. Then M is S-semicommutative.*

Proof. Let $f \in S$ and $m \in M$ with $f(m) = 0$. By hypothesis $g(m) \in mR$ for any $g \in S$. Then $fg(m) \in f(m)R = 0$. Hence $fg(m) = 0$ for all $g \in S$. So S is semicommutative. □

Lemma 2.13. *Let $M = M_1 \oplus M_2$. If M is weak duo (see* [8] *in detail) and M_1 and M_2 are S_1-semicommutative and S_2-semicommutative submodules respectively where* $\mathrm{End}(M_1) = S_1$ *and* $\mathrm{End}(M_2) = S_2$, *then M is S-semicommutative.*

Proof. Since M is weak duo, M_1 and M_2 are fully invariant submodules. Let $f(m) = 0$. If $m = m_1 + m_2$ where $m_1 \in M_1$ and $m_2 \in M_2$, then $f(m_1 + m_2) = 0$ and so $f(m_1) = f(m_2) = 0$. Since M_1 and M_2 are S_1- and S_2-semicommutative submodules respectively, for all $g \in S$, $fg(m) = 0$. □

S-semicommutative modules are not closed under direct sums. Let $R = \begin{pmatrix} F & F \\ F & F \end{pmatrix}$, where F is a field. It is well known that R is not a semicommutative ring and thus $F \bigoplus F$ is not an S-semicommutative module, since $\mathrm{End}_F(F \bigoplus F) \cong \begin{pmatrix} F & F \\ F & F \end{pmatrix}$. Also $F \bigoplus F$ is not an S-semicommutative module, but F is an S_1-semicommutative module where $S = \mathrm{End}_F(F \bigoplus F)$ and $S_1 = \mathrm{End}_F(F)$. Also we understand from this example that this property is not extension closed.

Now we investigate at least when this case can be possible?

Lemma 2.14. *Let R be a ring and I be a fully invariant reduced ideal of R. If R/I is an S-semicommutative ring where* $S = \mathrm{End}_R(R/I)$, *then R is an S_1-semicommutative where* $S_1 = \mathrm{End}_R(R)$.

Proof. Let $f(a) = 0$ where $f \in S_1$ and $g \in S_1$. Let $f_1 \colon R/I \to R/I$ and $g_1 \colon R/I \to R/I$ such that $f_1(r + I) = f(r) + I$ and $g_1(r + I) = g(r) + I$. Then f_1 and g_1 are module homomorphisms over R. Since R/I is an S-semicommutative ring $f_1g_1(a + I) =$

$fg(a) + I = I$ and so $fg(a) \in I$. Since $(aIf(1))^2 = 0$ and I is reduced, $aIf(1) = 0$. Then $(f(1)g(1)aI)^2 = f(1)g(1)(aIf(1))g(1)aI = 0$ and so $f(1)g(1)aI = 0$ and $f(g(a)) = 0$. Therefore R is an S_1-semicommutative module. □

Furthermore if R is a semicommutative ring and so an S-semicommutative module where $S = \mathrm{End}_R(R)$, then R/I may not be an S_1-semicommutative module where $S_1 = \mathrm{End}_R(R/I)$ and I is a right ideal. Let

$$R = \left\{ \begin{pmatrix} a & b & c & d \\ 0 & a & b & e \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix} : a, b, c, d, e \in F \right\} \quad \text{and} \quad I = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : a \in F \right\}$$

where F is a field. Let $f\left(\begin{pmatrix} a & b & c & d \\ 0 & a & b & e \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix} + I\right) = \begin{pmatrix} a & 0 & 0 & d \\ 0 & a & 0 & e \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} + I$ and $g\left(\begin{pmatrix} a & b & c & d \\ 0 & a & b & e \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix} + I\right) = \begin{pmatrix} a & 0 & 0 & e \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & o \end{pmatrix} + I$. Then $f, g \in S_1$ and $f\left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + I\right) = I$ and $f\left(g\left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + I\right)\right)$ $\neq I$ and so R/I is not an S_1-semicommutative module.

Lemma 2.15. *Let M be an S-semicommutative module. Consider the following*:

(1) *M is an S-Baer module.*
(2) *M is an S-quasi-Baer module.*
(3) *M is an S-p.q.-Baer module.*

Then (1) ⇔ (2) ⇒ (3).

Proof. (1)⇒(2)⇒(3) is clear. (2)⇒ (1). Let N be any submodule of M and $n \in N$. By hypothesis $l_S(n) = l_S(SnR)$. Hence $l_S(N) = l_S(SN)$. Since SN is a fully invariant submodule of M, by (2) $l_S(SN) = Se$ for some $e^2 = e \in S$. This completes the proof. □

PROPOSITION 2.16

Following are equivalent for an S-semicommutative module M.

(1) *M is an S-p.q.-Baer module.*
(2) *The left annihilator in S of any finitely generated R-submodule of M is generated* (*as a left ideal*) *by an idempotent of S.*

Proof. (2)⇒(1) is clear. (1)⇒(2): Assume that M is an S-p.q.-Baer module and let N be a finitely generated R-submodule of M. We will prove only for $n = 2$. Same proof will work for any n. Let $N = n_1R + n_2R$. By (1), $l_S(n_1R) = Se_1$ and $l_S(n_2R) = Se_2$. By Lemma 2.1, $e_1e_2 = e_2e_1$ and so e_1e_2 becomes an idempotent. Hence $l_S(N) = Se_1e_2$. This completes the proof. □

A module M_R is called a *principally projective* (or simply p.p.-module) if, for any $m \in M, r_R(m) = eR$ where $e^2 = e \in R$ (see [6]). In [6] Lee-Zhou introduced the following notation. For a module M_R, we consider $M[x] = \left\{\sum_{i=0}^{s} m_i x^i : s \geq 0, m_i \in M\right\}$, $M[x]$ is an Abelian group under an obvious addition operation. Moreover $M[x]$ becomes a right $R[x]$-module under the following scalar product operation:

For

$$m(x)=\sum_{i=0}^{s} m_i x^i \in M[x] \quad \text{and} \quad f(x)=\sum_{i=0}^{t} a_i x^i \in R[x],$$

$$m(x)f(x)=\sum_{k=0}^{s+t}\left(\sum_{i+j=k} m_i a_j\right)x^k.$$

By these operations $M[x]$ becomes a right module over $R[x]$. Similarly, $M[x]$ is a left $S[x]$-module by the scalar product:

For

$$m(x)=\sum_{i=0}^{s} m_i x^i \in M[x] \quad \text{and} \quad \alpha(x)=\sum_{i=0}^{t} f_i x^i \in S[x],$$

$$\alpha(x)m(x)=\sum_{k=0}^{s+t}\left(\sum_{i+j=k} f_i m_j\right)x^k.$$

PROPOSITION 2.17

Let M be an S-p.q.-Baer module. Then M is an S-semicommutative module if and only if $fem = efm$, for any $m \in M$, $f \in S$, and $e^2 = e \in S$.

Proof. The necessity is clear from Lemma 2.1. Conversely, assume that $fem = efm$, for any $m \in M$, $f \in S$ and $e^2 = e \in S$. Let $fm = 0$ for some $f \in S$ and $m \in M$. There exists $e^2 = e \in S$ such that $f \in l_S(m) = Se$. Then $f = fe$ and $em = 0$. For any $g \in S$, by assumption $fgm = fegm = fgem = 0$. Hence M is S-semicommutative. □

Lemma 2.18. *Let M be a module and $S = \text{End}(M)$. If M is an S-p.q.-Baer module, then M is S-semicommutative if and only if $M[x]$ is $S[x]$-semicommutative.*

Proof. Assume that M is S-semicommutative module. Let $m(x) = \sum m_i x^i \in M[x]$, $f(x) = \sum f_j x^j \in S[x]$ satisfy $f(x)m(x) = 0$. Then

$$f_0 m_0 = 0, \tag{1}$$

$$f_0 m_1 + f_1 m_0 = 0, \tag{2}$$

$$f_0 m_2 + f_1 m_1 + f_2 m_0 = 0, \tag{3}$$

$$\cdots . \tag{4}$$

Let $l_S(m_0) = Se_0$, $l_S(m_1) = Se_1$, $l_S(m_2) = Se_2, \ldots$ where $e_i^2 = e_i \in S$. By hypothesis S is abelian and by (1), $f_0 e_0 = e_0 f_0 = f_0$. Left multiply (2) by e_0 to obtain $f_0 m_1 = 0$. Hence $f_1 m_0 = 0$. So $f_0 e_1 = e_1 f_0 = f_0$ and $f_1 e_0 = e_0 f_1 = f_1$. Left multiply (3) by e_0 to obtain $f_2 m_0 = 0$ so (3) becomes $f_0 m_2 + f_1 m_1 = 0$. Multiply this equality by e_1 from left to have $f_2 m_0 = 0$. Hence $f_1 m_1 = 0$. Continuing in this way we may obtain $f_i m_j = 0$ for all i and j. The rest is clear. □

To get rid of confusions we recall that $M[x]$ is an $S[x]$-p.q.-*Baer module* if for any $m(x) \in M[x]$, there exists $e^2 = e \in S[x]$ such that $l_{S[x]}(m(x)) = eS[x]$, and $M[x]$ is an $S[x]$-Baer-module if for any $R[x]$ - submodule A of $M[x]$, there exists $e^2 = e \in S[x]$ such that $l_{S[x]}(A) = eS[x]$, and $M[x]$ is an $S[x]$-q.-Baer module if for any fully invariant $R[x]$-submodule A of $M[x]$, there exists $e^2 = e \in S[x]$ such that $l_{S[x]}m(x) = eS[x]$.

Lemma 2.19. *Let M be a module such that $S = \text{End}(M)$ is a semicommutative ring. Then*

(1) *Every idempotent of $S[x]$ is in S and $S[x]$ is abelian.*
(2) *Every idempotent of $S[[x]]$ is in the S and $S[[x]]$ is abelian.*

Proof. Clear from Lemma 8 of [5].

Theorem 2.20. *Let M be an S-semicommutative module. Then*

(1) *M is an S-p.q.-Baer module if and only if $M[x]$ is an $S[x]$-p.q.-Baer module.*
(2) *M is an S-Baer module if and only if $M[x]$ is an $S[x]$-Baer module.*
(3) *M is an S-q.-Baer module if and only if $M[x]$ is an $S[x]$-q.-Baer module.*

Proof. Let M be an S-semicommutative module. By Lemma 2.1, S is semicommutative and so an abelian ring.

(1) $\Rightarrow$. Assume that M is an S-p.q.-Baer module. Let $m(x) = \sum_{i=0}^{k} m_i x^i \in M[x]$, $f(x) = \sum_{j=0}^{t} f_j x^j \in S[x]$ satisfy $f(x)m(x) = 0$. Let $l_S(m_i) = Se_i$ where $e_i^2 = e_i \in S(i = 0, 1, 2, \ldots, k)$. Since S is abelian, $f_i m_j = 0$ implies $f_i e_j = e_j f_i = f_i$ for all $i = 0, 1, 2, \ldots, t$ and $j = 0, 1, 2, \ldots, k$. Let $e = e_0 e_1 e_2 \ldots e_k$. Then e is a central idempotent in S. We prove $l_{S[x]}(m(x)) = S[x]e$. Let $f(x) = \sum f_j x^j \in l_{S[x]}(m(x))$, then $f_j e = f_j$ and so $f(x)e = f(x)$. Hence $f(x) \in S[x]e$ and so $l_{S[x]}(m(x)) \leq S[x]e$. Let $g(x) \in S[x]e$. Since S is abelian, $em(x) = 0$ and $g(x)em(x) = 0$. Hence $S[x]e \leq l_{S[x]}(m(x))$.

$\Leftarrow$. Suppose that $M[x]$ is an $S[x]$-p.q.-Baer module. Let $m \in M$. Then $l_{S[x]}(m) = S[x]e$ for some $e^2 = e \in S[x]$. By Lemma 2.19, $e \in S$. Clearly $(S[x]e) \cap S = Se$. Hence $l_S(m) = Se$.

(2) $\Rightarrow$. Assume that M is an S-Baer module. Let A be any $R[x]$-submodule of $M[x]$. We will prove that there exists $e^2 = e \in S[x]$ such that $l_{S[x]}(A) = S[x]e$. Let A^* be the right R-submodule of M generated by the coefficients of elements of A. By assumption $l_S(A^*) = Se$ for some $e^2 = e \in S$. Then $S[x]e \leq l_{S[x]}(A)$ is clear. To prove reverse inclusion, let $g(x) = c_0 + c_1 x + \cdots + c_n \in l_{S[x]}(A)$. Then $g(x)A = 0$ and so $g_i A = 0$. By Lemma 2.1, S is semicommutative and so abelian. Hence S is Armendariz, that is, $g_i A^* = 0$, $g_i \in l_S(A^*) = Se$ and $g_i e = g_i$ for all $0 \leq i \leq n$. So $g(x)e = g(x) \in S[x]e$. $l_{S[x]}(A) \leq S[x]e$. Therefore $l_{S[x]}(A) = S[x]e$.

$\Leftarrow$. Assume that $M[x]$ is an $S[x]$-Baer-module. Let A be any submodule of M. Then $l_{S[x]}(A[x]) = S[x]e$ for some $e^2 = e \in S[x]$. By Lemma 2.19, $e \in S$. Then $(S[x]e) \cap S = Se$. Hence M is an S-Baer module.

(3) Similar to proof of (2). □

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