# ON POWER SERIES SUBSPACES OF CERTAIN NUCLEAR FRÉCHET SPACES

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ABSTRACT. The diametral dimension, $\Delta(E)$, and the approximate diametral dimension, $\delta(E)$ of an element $E$ of a large class of nuclear Fréchet spaces are set theoretically between the corresponding invariant of power series spaces $\Lambda_1(\varepsilon)$ and $\Lambda_\infty(\varepsilon)$ for some exponent sequence $\varepsilon$. Aytuna et al., [2], proved that $E$ contains a complemented subspace which is isomorphic to $\Lambda_\infty(\varepsilon)$ provided $\Delta(E) = \Delta(\Lambda_\infty(\varepsilon))$ and $\varepsilon$ is stable. In this article, we will consider the other extreme case and we proved that in this large family, there exist nuclear Fréchet spaces, even regular nuclear Köthe spaces, satisfying $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$ such that there is no subspace of $E$ which is isomorphic to $\Lambda_1(\varepsilon)$.

## 1. INTRODUCTION

Fréchet spaces are one of the leading class of locally convex spaces and include most of the important examples of non-normable locally convex spaces. Power series spaces constitute a well studied class in the theory of Fréchet spaces. Subspaces and quotient spaces of a nuclear stable power series space are characterized by Vogt and Wagner ([15], [17], [18]) in terms of diametral dimension and DN-$\Omega$ type linear-topological invariants. The topological invariants $\underline{\text{DN}}$ and $\Omega$ are enjoyed by many natural nuclear Fréchet spaces appearing in analysis and these invariants play an important role in this study.

Let $E$ be a nuclear Fréchet space which satisfies $\underline{DN}$ and $\Omega$. Then it is a known fact that the diametral dimension $\Delta(E)$ and the approximate diametral dimension $\delta(E)$ of $E$ are set theoretically between corresponding invariant of power series spaces $\Lambda_1(\varepsilon)$ and $\Lambda_\infty(\varepsilon)$ for some specific exponent sequence $\varepsilon$. The sequence $\varepsilon$ is called associated exponent sequence of E. In [2], Aytuna et al. proved that a nuclear Fréchet space $E$ with the properties $\underline{DN}$ and $\Omega$ contains a complemented copy of $\Lambda_\infty(\varepsilon)$ provided the diametral dimensions of $E$ and $\Lambda_\infty(\varepsilon)$ are equal and $\varepsilon$ is stable. In this article, we deal with the other extreme, namely, the main question in this article is:

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**Question 1.1.** *Let $E$ be a nuclear Fréchet space with the properties $\underline{DN}$ and $\Omega$ and $\varepsilon$ be the associated exponent sequence of $E$. Is there a (complemented) subspace of $E$ which is isomorphic to $\Lambda_1(\varepsilon)$ if $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$?*

This problem led us to examine the relationship between the diametral dimension and the other invariants. The most appropriate topological invariants for comparison with the diametral dimension is the approximate diametral dimension. Then, we ask the following question:

**Question 1.2.** *Let $E$ be a nuclear Fréchet space with the properties $\underline{DN}$ and $\Omega$. If diametral dimension of $E$ coincides with that of a power series space, then does this imply that the approximate diametral dimension also do the same and vice versa?*

In [5], we showed that Question 1.2 has an affirmative answer when power series space is of infinite type. Then we searched an answer for the Question 1.2 in the finite type case and, in this regard, we first proved that the condition $\delta(E) = \delta(\Lambda_1(\varepsilon))$ always implies $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$. We also constructed some sufficient conditions to prove the other direction. It turns out that the existence of a prominent bounded subset in the nuclear Fréchet space $E$ plays a decisive role for the answer of Question 1.2. In [5, Theorem 4.8], we proved that $\delta(E) = \delta(\Lambda_1(\varepsilon))$ if and only if $E$ has a prominent bounded set and $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$.

In this article, after giving some preliminary materials in Section 2, we construct a family $\mathcal{K}$ of nuclear Köthe spaces $K(a_{k,n})$ parametrized by a sequence $\alpha$ satisfying the properties $\underline{DN}$ and $\Omega$. First we show that for an element of the family of $\mathcal{K}$ which is parameterized by a stable sequence $\alpha$, $\Delta(K(a_{k,n})) = \Delta(\Lambda_1(\alpha))$ and $\delta(K(a_{k,n})) = \delta(\Lambda_1(\alpha))$. Second, we prove that for any element of the family of $\mathcal{K}$ which is parameterized by an unstable sequence $\alpha$, $\Delta(K(a_{k,n})) = \Delta(\Lambda_1(\varepsilon))$ and $\delta(K(a_{k,n})) \neq \delta(\Lambda_1(\varepsilon))$ for its associated exponent sequence $\varepsilon$. This show that the Question 1.2 has a negative answer for power series space of finite type. Furthermore, we prove in Theorem 4.1 that the first question has a negative answer, that is, $\Lambda_1(\varepsilon)$ is not isomorphic to any subspace of these Köthe spaces $K(a_{k,n})$, let alone is isomorphic to a complemented subspace, though the condition $\Delta(K(a_{k,n})) = \Delta(\Lambda_1(\varepsilon))$ is satisfied. Motivated by our finding in [5], we compile some additional information, for instance, for any element $E$ of the family $\mathcal{K}$ parameterized by an unstable sequence,

1. $E$ does not have a prominent bounded set.

2. Although the equality $\Delta(E) = \Lambda_1(\varepsilon)$ is satisfied and the canonical imbedding from $\Delta(E)$ into $\Lambda_1(\varepsilon)$ has a closed graph, the canonical imbedding from $\Delta(E)$ into $\Lambda_1(\varepsilon)$ is not continuous.

## 2. PRELIMINARIES

In this section, after establishing terminology and notation, we collect some basic facts and definitions that are needed them in the sequel.

Throughout the article, $E$ will denote a nuclear Fréchet space with an increasing sequence of Hilbertian seminorms $(\|.\|_k)_{k\in\mathbb{N}}$. For a Fréchet space $E$, we will denote the class of all neighborhoods of zero in $E$ and the class of all bounded sets in $E$ by $\mathcal{U}(E)$ and $\mathcal{B}(E)$, respectively. If $U$ and $V$ are absolutely convex sets of $E$ and $U$ absorbs $V$, that is, $V \subseteq CU$ for some $C > 0$, and $L$ is a subspace of $E$, then we set;

$$\delta(V, U, L) = \inf\{t > 0 : V \subseteq tU + L\}.$$

The $n^{th}$ *Kolmogorov diameter* of $V$ with respect to $U$ is defined as;

$$d_n(V, U) = \inf\{\delta(V, U, L) : \dim L \leq n\} \qquad n = 0, 1, 2, ...$$

Let $U_1 \supset U_2 \supset \cdots \supset U_p \supset \cdots$ be a base of neighborhoods of zero of Fréchet space E. The diametral dimension of $E$ is defined as

$$\Delta(E) = \left\{(t_n)_{n\in\mathbb{N}} : \forall p \in \mathbb{N}\ \exists\, q > p \ \lim_{n\to\infty} t_n d_n(U_q, U_p) = 0\right\}.$$

Demeulenaere et al. [4] showed that the diametral dimension of a nuclear Fréchet space can also be represented as

$$\Delta(E) = \left\{(t_n)_{n\in\mathbb{N}} : \forall\, p \in \mathbb{N}\ \exists\, q > p \ \sup_{n\in\mathbb{N}} |t_n|\, d_n(U_q, U_p) < +\infty\right\}.$$

The approximate diametral dimension of a Fréchet space $E$ is defined as

$$\delta(E) = \left\{(t_n)_{n\in\mathbb{N}} : \exists\, U \in \mathcal{U}(E)\ \exists\, B \in \mathcal{B}(E) \ \lim_{n\to\infty} \frac{t_n}{d_n(B, U)} = 0\right\}.$$

It follows from Proposition 6.6.5 of [11] that for a Fréchet space $E$ with the base of neighborhoods $U_1 \supset U_2 \supset \cdots \supset U_p \supset \cdots$, the approximate diametral dimension can be represented as;

$$\delta(E) = \left\{(t_n)_{n\in\mathbb{N}} : \exists p \in \mathbb{N}\ \forall\, q > p \ \lim_{n\to\infty} \frac{t_n}{d_n(U_q, U_p)} = 0\right\}.$$

The following proposition shows how the diametral dimension and the approximate diametral dimension passes into subspaces:

**Proposition 2.1.** *Let $E$ be a Fréchet space and $F$ be a subspace or a quotient of $E$. Then,*
*1.* $\Delta(E) \subseteq \Delta(F)$.

*2.* $\delta(F) \subseteq \delta(E)$.
*Hence the diametral dimension and the approximate diametral dimension are linear topological invariants.*

*Proof.* [11, Proposition 6.6.7 and Proposition 6.6.25] □

A matrix $(a_{k,n})_{k,n\in\mathbb{N}}$ of non-negative numbers is called a *Köthe matrix* if it is satisfies that for each $k \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ with $a_{k,n} > 0$ and $a_{k,n} \leq a_{k,n+1}$ for all $k, n \in \mathbb{N}$. For a Köthe matrix $(a_{k,n})_{k,n\in\mathbb{N}}$,

$$K(a_{k,n}) = \left\{ x = (x_n) : \|x\|_k := \sum_{n=1}^{\infty} |x_n| \, a_{k,n} < +\infty \text{ for all } k \in \mathbb{N} \right\}$$

is called a *Köthe space*. Every Köthe space is a Fréchet space given by the semi-norms in its definition. Nuclearity of a Köthe space was characterized as follows:

**Theorem 2.2.** *[Grothendieck-Pietsch] $K(a_{kn})$ is nuclear Köthe space if and only if for every $k \in \mathbb{N}$, there exists a $l > k$ so that $\displaystyle\sum_{n=1}^{\infty} \frac{a_{k,n}}{a_{l,n}} < +\infty$.*

*Proof.* [8, Theorem 28.15]. □

Dynin-Mitiagin Theorem [8, Theorem 28.12] states that if a nuclear Fréchet space E with the sequence of seminorms $(\|.\|_k)_{k\in\mathbb{N}}$ has a Schauder basis $(e_n)_{n\in\mathbb{N}}$, then it is canonically isomorphic to a nuclear Köthe space defined by the matrix $(\|e_n\|_k)_{k,n\in\mathbb{N}}$. Therefore, it is important to understand the structure of nuclear Köthe spaces in the theory of nuclear Fréchet spaces.

Terzioğlu gave an estimation for $n^{th}$-Kolmogorov diameters of a Köthe space $K(a_{k,n})$ by using the matrix $(a_{k,n})_{k,n\in\mathbb{N}}$.

**Proposition 2.3.** *Let $K(a_{k,n})$ be a Köthe space and fixed $n \in \mathbb{N}$. Assume $J \subset \mathbb{N}$ with $|J| = n+1$ and $I \subset \mathbb{N}$ with $|I| \leq n$. Then for every $p$ and $q > p$,*

$$\inf\left\{ \frac{a_{p,i}}{a_{q,i}} : i \in J \right\} \leq d_n(U_q, U_p) \leq \sup\left\{ \frac{a_{p,i}}{a_{q,i}} : i \notin I \right\}.$$

*Proof.* [12, Proposition 1]. □

**Definition 2.4.** *A Köthe space $K(a_{k,n})$ is called regular if the inequality $\dfrac{a_{k+1,n}}{a_{k,n}} \leq \dfrac{a_{k+1,n+1}}{a_{k,n+1}}$ is satisfied for all $k, n \in \mathbb{N}$*

**Remark 2.5.** *In the light of the above proposition, we conclude that for any regular Köthe space $K(a_{p,n})$, the $n^{th}$-Kolmogorov diameter is $d_n(U_q, U_p) = \dfrac{a_{p,n+1}}{a_{q,n+1}}$. If, on the other hand, $K(a_{p,n})$ is not regular, then, one can find*

*Kolmogorov diameters by rewriting the sequence* $\left(\dfrac{a_{p,n}}{a_{q,n}}\right)_{n\in\mathbb{N}}$ *with terms in a descending order so that the* $n^{th}$*-Kolmogorov diameter of* $K(a_{p,n})$ *is nothing but the* $n+1-th$ *term of this descending sequence.*

Power series spaces are the most important family of Köthe spaces and they have a significant role in this work, for a comprehensive survey see [13]. Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a non-negative increasing sequence with $\lim_{n\to\infty} \alpha_n = +\infty$. A power series space of finite type is defined by

$$\Lambda_1(\alpha) := \left\{ x = (x_n)_{n\in\mathbb{N}} : \|x\|_k = \sum_{n=1}^{\infty} |x_n|\, e^{-\frac{1}{k}\alpha_n} < +\infty \text{ for all } k \in \mathbb{N} \right\}$$

and a power series space of infinite type is defined by

$$\Lambda_\infty(\alpha) := \left\{ x = (x_n)_{n\in\mathbb{N}} : \|x\|_k = \sum_{n=1}^{\infty} |x_n|\, e^{k\alpha_n} < +\infty \text{ for all } k \in \mathbb{N} \right\}.$$

The nuclearity of a power series space of finite type $\Lambda_1(\alpha)$ and of infinite type $\Lambda_\infty(\alpha)$ are equivalent to the conditions $\lim_{n\to\infty} \dfrac{\ln(n)}{\alpha_n} = 0$ and $\sup_{n\in\mathbb{N}} \dfrac{\ln(n)}{\alpha_n} < +\infty$, respectively.

**Definition 2.6.** *An exponent sequence* $\alpha$ *is called finitely nuclear if* $\Lambda_1(\alpha)$ *is nuclear.*

Diametral dimension and approximate diametral dimension of power series spaces are $\Delta(\Lambda_1(\alpha)) = \Lambda_1(\alpha)$, $\Delta(\Lambda_\infty(\alpha)) = \Lambda_\infty(\alpha)'$, $\delta(\Lambda_1(\alpha)) = \Lambda_1(\alpha)'$ and $\delta(\Lambda_\infty(\alpha)) = \Lambda_\infty(\alpha)$ for details see [3] and [9].

An exponent sequence $\alpha$ is called

$$\begin{array}{lll} \textit{stable} & \text{if} & \sup_{n\in\mathbb{N}} \dfrac{\alpha_{2n}}{\alpha_n} < +\infty, \\[2ex] \textit{weakly-stable} & \text{if} & \sup_{n\in\mathbb{N}} \dfrac{\alpha_{n+1}}{\alpha_n} < +\infty, \\[2ex] \textit{unstable} & \text{if} & \lim_{n\to\infty} \dfrac{\alpha_{n+1}}{\alpha_n} = +\infty. \end{array}$$

It follows that $\alpha$ is stable, respectively weakly-stable, if and only if $E \cong E \times E$, respectively, $E \cong E \times \mathbb{K}$ where $E = \Lambda_r(\alpha)$ for $r = 1$ or $r = \infty$, for proofs see [6].

Subspaces and quotient spaces of a nuclear stable power series space are characterized by Vogt and Wagner ([15],[17], [18]) in terms of diametral dimension and DN-$\Omega$ type linear-topological invariants. The topological invariants $\underline{\text{DN}}$ and $\Omega$ are enjoyed by many natural nuclear Fréchet spaces appearing in analysis and these invariants play an important role in this study.

**Definition 2.7.** *A Fréchet space* $(E, \|.\|_k)_{k\in\mathbb{N}}$ *is said to have the property:*

***($\underline{DN}$)*** $\exists\, k \;\; \forall\, j \;\; \exists\, l,\; C > 0,\; 0 < \lambda < 1$

$$\|x\|_j \leq C\, \|x\|_k^{\lambda}\, \|x\|_l^{1-\lambda} \qquad \forall\, x \in E$$

**($\Omega$)** $\forall\, p \;\; \exists\, q \;\; \forall\, k \;\; \exists\, C > 0,\; 0 < \tau < 1$

$$\|y\|_q^{*} \leq C \|y\|_p^{*1-\theta} \|y\|_k^{*\theta} \qquad \forall\, y \in E^{'}$$

*where* $\|y\|_k^{*} := \sup\{|y(x)| : \|x\|_k \leq 1\} \in \mathbb{R} \cup \{+\infty\}$ *is the gauge functional of the polar* $U_k^{\circ}$ *for* $U_k = \{x \in E : \|x\|_k \leq 1\}$.

In [16], D. Vogt characterized $\Omega$ for Köthe spaces in terms of Köthe matrix as follows:

**Proposition 2.8.** *A Köthe space* $K(a_{k,n})$ *has the property* $\Omega$ *if and only if the condition*

$$\forall\, p \;\; \exists\, q \;\; \forall\, k \;\; \exists\, j > 0,\; C > 0 \qquad (a_{p,n})^j\, a_{k,n} \leq C\, (a_{q,n})^{j+1} \qquad \forall n \in \mathbb{N}$$

*is satisfied.*

*Proof.* [16, Proposition 5.3]. □

By using the technique in [16, 5. 1 Proposition], one can easily obtain the following:

**Proposition 2.9.** *A Köthe space* $K(a_{k,n})$ *has the property* $\underline{DN}$ *if and only if the condition*

$$\exists\, p_0 \;\; \forall\, p \;\; \exists\, q \;\; \exists\, 0 < \lambda < 1,\; C > 0 \qquad a_{p,n} \leq C\, (a_{p_0,n})^{\lambda}\, (a_{q,n})^{1-\lambda} \qquad \forall n \in \mathbb{N}$$

*is satisfied.*

Now we give the important result which gives a relation between the diametral dimension/approximate diametral dimension of a nuclear Fréchet spaces with the properties $\underline{DN}$, $\Omega$ and that of a power series spaces $\Lambda_1(\varepsilon)$ and $\Lambda_\infty(\varepsilon)$ for some special exponent sequence $\varepsilon$.

**Proposition 2.10.** *Let* $E$ *be a nuclear Fréchet space with the properties* $\underline{DN}$ *and* $\Omega$. *There exists an exponent sequence (unique up to equivalence)* $\varepsilon = (\varepsilon_n)$ *satisfying:*

$$\Delta(\Lambda_1(\varepsilon)) \subseteq \Delta(E) \subseteq \Delta(\Lambda_\infty(\varepsilon)). \tag{2.1}$$

*Furthermore,* $\Lambda_1(\alpha) \subseteq \Delta(E)$ *implies* $\Lambda_1(\alpha) \subseteq \Lambda_1(\varepsilon)$ *and* $\Delta(E) \subseteq \Lambda'_\infty(\alpha)$ *implies* $\Lambda'_\infty(\varepsilon) \subseteq \Lambda'_\infty(\alpha)$.

*Proof.* [2, Proposition 1.1]. □

**Definition 2.11.** *Let $E$ be a nuclear Fréchet space with the properties $\underline{DN}$ and $\Omega$. The sequence $\varepsilon$ (unique up to equivalence) in the above proposition is called the* ***associated exponent sequence*** *of $E$ in* [2].

We note that $\Lambda_\infty(\varepsilon)$ is always nuclear provided $E$ is nuclear, but it may happen that $\Lambda_1(\varepsilon)$ is not nuclear. For example, if we take the space of rapidly decreasing sequence $s = \Lambda_\infty(\ln(n))$, the associated exponent sequence of $s$ is $(\ln(n))_{n\in\mathbb{N}}$ and $\Lambda_1(\ln(n))$ is not nuclear.

In the proof of the above proposition, Aytuna et al. showed that there exists an exponent sequence (unique up to equivalence) $(\varepsilon_n)$ such that for each $p \in \mathbb{N}$ and $q > p$, there exist $C_1, C_2 > 0$ and $a_1, a_2 > 0$ satisfying

$$C_1\, e^{-a_1\,\varepsilon_n} \leq d_n\,(U_q, U_p) \leq C_2\, e^{-a_2\,\varepsilon_n}$$

for all $n \in \mathbb{N}$. From this inequality, one can easily obtain

$$\delta\,(\Lambda_\infty\,(\varepsilon)) \subseteq \delta\,(E) \subseteq \delta\,(\Lambda_1\,(\varepsilon))\,.$$

For a nuclear Fréchet space $E$ with the properties $\underline{DN}$ and $\Omega$ and the associated exponent sequence $\varepsilon$, concidence of the diametral dimension of $E$ with that of power series spaces defined by $\varepsilon$ form two extreme cases. The extreme case $\Delta(E) = \Delta\,(\Lambda_\infty(\varepsilon))$ gives an information about a (complemented) subspace of a nuclear Fréchet space $E$ with the properties $\underline{DN}$ and $\Omega$ and stable associated exponent sequence $\varepsilon$. In [2], Aytuna et al. proved that a nuclear Fréchet space $E$ with the properties $\underline{DN}$ and $\Omega$ contains a complemented copy of $\Lambda_\infty(\varepsilon)$ provided that $\Delta(E) = \Delta(\Lambda_\infty(\varepsilon))$ and $\varepsilon$ is stable.

**Theorem 2.12.** *Let $E$ be a nuclear Fréchet space with the properties $\underline{DN}$ and $\Omega$ and stable associated exponent sequence $\varepsilon$. If $\Delta\,(E) = \Delta\,(\Lambda_\infty\,(\varepsilon))$, then $E$ has complemented subspace which is isomorphic to $\Lambda_\infty\,(\varepsilon)$.*

*Proof.* [2, Theorem 1.2]. □

On the other hand, there is no information for the other extreme $\Delta(E) = \Delta\,(\Lambda_1(\varepsilon))$. This leads to ask the Question 1.1 in Introduction. We need the following proposition characterizing coincidence of $\delta(E)$ with $\delta(\Lambda_1(\varepsilon))$ given by A. Aytuna in [1]:

**Proposition 2.13.** *Let $E$ be a nuclear Fréchet space $E$ with the properties $\underline{DN}$, $\Omega$ and associated exponent sequence $\varepsilon$. Then*

$$\delta\,(E) = \delta\,(\Lambda_1\,(\varepsilon)) \qquad \Leftrightarrow \qquad \inf_{p}\sup_{q\geq p}\limsup_{n\in\mathbb{N}} \frac{\varepsilon_n\,(p,q)}{\varepsilon_n} = 0$$

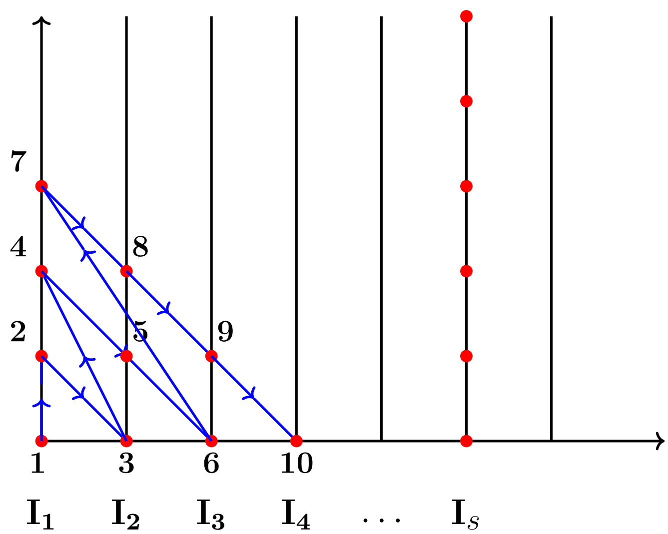
*where* $\varepsilon_n(p,q) = -\log d_n(U_q, U_p)$.

*Proof.* [1, Corollary 1.10] □

## 3. $\mathcal{K}_\alpha$ Spaces

In this section, we will construct a family of nuclear Köthe spaces with the properties $\underline{DN}$ and $\Omega$ and parameterized by a finitely nuclear sequence $\alpha$ and show that a subfamily of these Köthe spaces satisfied that $\Delta(K(a_{k,n})) = \Delta(\Lambda_1(\varepsilon))$ and $\delta(K(a_{k,n})) \neq \delta(\Lambda_1(\varepsilon))$ for its associated exponent sequence $\varepsilon$. This shows that Question **1.2** has a negative answer.

We proceed as follows: First, we divide natural numbers $\mathbb{N}$ into infinite disjoint union of infinite subsets. For this purpose, we order the elements of $\mathbb{N}^2$ by matching them with the elements of $\mathbb{N}$ such that any element $(x,y) \in \mathbb{N}^2$ corresponds to the element $\frac{(x+1)(x+2)}{2} + y(x+1) + \frac{y(y-1)}{2} \in \mathbb{N}$. One can visualize this ordering as shown in the following diagram:




As shown in the above diagram, each vertical line $I_s$ has infinitely many elements and $\mathbb{N}$ can be expressed as an infinite disjoint union of $I_s$, that is, $N = \bigcup_{s\in\mathbb{N}} I_s$.

**Definition 3.1.** *Let* $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ *be a strictly increasing, positive, finitely nuclear sequence. We define a matrix* $(a_{k,n})_{k,n\in\mathbb{N}}$ *by setting:*

$$a_{k,n} = \begin{cases} e^{-\frac{1}{k}\alpha_n}, & \text{if } k \leq s \\ e^{\left(-\frac{1}{k}+1\right)\alpha_n}, & \text{if } k \geq s+1. \end{cases} \tag{3.1}$$

*where $n \in I_s,\ s \in \mathbb{N}$.*

Infact, $(a_{k,n})_{k,n\in\mathbb{N}}$ is a Köthe matrix, since for every $n, k \in \mathbb{N},\ 0 < a_{k,n} \leq a_{k+1,n}$. We denote the Köthe space generated by a matrix $(a_{k,n})_{k,n\in\mathbb{N}}$ as in 3.1 by $\mathcal{K}_\alpha$. We say that the space $\mathcal{K}_\alpha$ is parameterized by the sequence $\alpha$. We denote the family of all Köthe space $\mathcal{K}_\alpha$ by $\mathcal{K}$. Now, we show that each element of the family $\mathcal{K}$ is nuclear and satisfies the properties $\underline{DN}$ and $\Omega$:

**Lemma 3.2.** *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ parametrized by $\alpha = (\alpha_n)_{n\in\mathbb{N}}$. Then, $\mathcal{K}_\alpha$ is nuclear and has the properties $\underline{DN}$ and $\Omega$.*

*Proof.* For the nuclearity of $\mathcal{K}_\alpha$, we show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{a_{k,n}}{a_{k+1,n}}$ is convergent for each $k \in \mathbb{N}$. Since $\dfrac{a_{k,n}}{a_{k+1,n}} \leq e^{\left(-\frac{1}{k}+\frac{1}{k+1}\right)\alpha_n}$ for every $k, n \in \mathbb{N}$ and $\Lambda_1(\alpha)$ is nuclear, then the series $\displaystyle\sum_{n=1}^{\infty} \frac{a_{k,n}}{a_{k+1,n}}$ is convergent. By Theorem 2.2, $\mathcal{K}_\alpha$ is nuclear, as asserted.

We now prove that $\mathcal{K}_\alpha$ has the $\underline{DN}$ property by using Proposition 2.9. We will show that for all $p \in \mathbb{N}$ there exists a $0 < \lambda < 1$ such that the inequality

$$a_{p,n} \leq (a_{1,n})^{\lambda} (a_{p+1,n})^{1-\lambda} \tag{3.2}$$

is satisfied for all $n \in \mathbb{N}$. Let $p, n \in \mathbb{N}$ and assume $n \in I_s,\ s \in \mathbb{N}$. There are two cases for $p$ and $s$: $p \leq s$ or $p > s$. First we assume that $p \leq s$: In this case, $a_{1,n} = e^{-\alpha_n}$, $a_{p,n} = e^{-\frac{1}{p}\alpha_n}$ and $a_{p+1,n} \geq e^{-\frac{1}{p+1}\alpha_n}$. Then, the inequality 3.2 is satisfied for any $\lambda < \dfrac{\frac{1}{p}-\frac{1}{p+1}}{1-\frac{1}{p+1}}$. Second we assume that $s < p$: In this case, $a_{1,n} = e^{-\alpha_n}$, $a_{p,n} = e^{\left(-\frac{1}{p}+1\right)\alpha_n}$ and $a_{p+1,n} = e^{\left(-\frac{1}{p+1}+1\right)\alpha_n}$. But then the inequality 3.2 is satisfied for any $\lambda < \dfrac{\frac{1}{p}-\frac{1}{p+1}}{2-\frac{1}{p+1}}$. Hence, if we choose a $\lambda > 0$ satisfying

$$\lambda < \min\left\{\frac{\frac{1}{p}-\frac{1}{p+1}}{1-\frac{1}{p+1}}, \frac{\frac{1}{p}-\frac{1}{p+1}}{2-\frac{1}{p+1}}\right\} = \frac{\frac{1}{p}-\frac{1}{p+1}}{2-\frac{1}{p+1}}$$

then inequality 3.2 holds in general and so $\mathcal{K}_\alpha$ has the property $\underline{\text{DN}}$, as claimed.

We now prove that $\mathcal{K}_\alpha$ has $\Omega$ by using Proposition 2.8. We will show that for all $p \in \mathbb{N}$ and $k > p$ there exists a $j > 0$ such the inequality

$$(a_{p,n})^{j}\, a_{k,n} \leq (a_{p+1,n})^{j+1} \tag{3.3}$$

is satisfied for all $n \in \mathbb{N}$. Let $p, n \in \mathbb{N}$ and assume $n \in I_s$, $s \in \mathbb{N}$. There are two case for $p$ and $s$: $p \leq s$ or $p > s$. First we assume that $p \leq s$: In this case, $a_{p,n} = e^{-\frac{1}{p}\alpha_n}$, $a_{p+1,n} \geq e^{-\frac{1}{p+1}\alpha_n}$ and $a_{k,n} \leq e^{\left(-\frac{1}{k}+1\right)\alpha_n}$ for all $k \geq p$. Then, the inequality 3.3 is satisfied for any $j \geq \dfrac{\frac{1}{p+1} - \frac{1}{k} + 1}{\frac{1}{p} - \frac{1}{p+1}}$. Second we assume that $s < p$: In this case, $a_{p,n} = e^{\left(-\frac{1}{p}+1\right)\alpha_n}$, $a_{p+1,n} = e^{\left(-\frac{1}{p+1}+1\right)\alpha_n}$ and $a_{k,n} = e^{\left(-\frac{1}{k}+1\right)\alpha_n}$ for all $k \geq p$. Therefore, the inequality 3.3 is satisfied for any $j \geq \dfrac{\frac{1}{p+1} - \frac{1}{k}}{\frac{1}{p} - \frac{1}{p+1}}$. Now, we choose a $j > 0$ satisfying

$$j \geq \max\left(\frac{\frac{1}{p+1} - \frac{1}{k} + 1}{\frac{1}{p} - \frac{1}{p+1}}, \frac{\frac{1}{p+1} - \frac{1}{k}}{\frac{1}{p} - \frac{1}{p+1}}\right) = \frac{\frac{1}{p+1} - \frac{1}{k} + 1}{\frac{1}{p} - \frac{1}{p+1}}$$

and so that the inequality 3.3 is satisfied for all $n \in \mathbb{N}$. Hence $\mathcal{K}_\alpha$ has the property $\Omega$, as claimed. □

**Remark 3.3.** *It is worth noting that any element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ does not have the property $(d_2)$,*

$$(d_2): \qquad \forall k \quad \exists j \quad \forall l \qquad \sup_n \frac{a_{kn}\, a_{ln}}{(a_{jn})^2} < +\infty.$$

*Since for all $j \in \mathbb{N}$, $n \in I_j$, $a_{1,n} = e^{-\alpha_n}$, $a_{jn} = e^{-\frac{1}{j}\alpha_n}$, $a_{j+1,n} = e^{\left(-\frac{1}{j+1}+1\right)\alpha_n}$,*

$$\frac{a_{1,n}\, a_{j+1,n}}{(a_{jn})^2} = e^{\frac{j+2}{j(j+1)}\alpha_n} \qquad \text{and} \qquad \sup_{n \in I_j} \frac{a_{1,n}\, a_{j+1,n}}{(a_{jn})^2} = \sup_{n \in \mathbb{N}} \frac{a_{1,n}\, a_{j+1,n}}{(a_{jn})^2} = +\infty$$

*then $\mathcal{K}_\alpha$ does not have the property $(d_2)$. So the family $\mathcal{K}$ does not contain a power series space of finite type.*

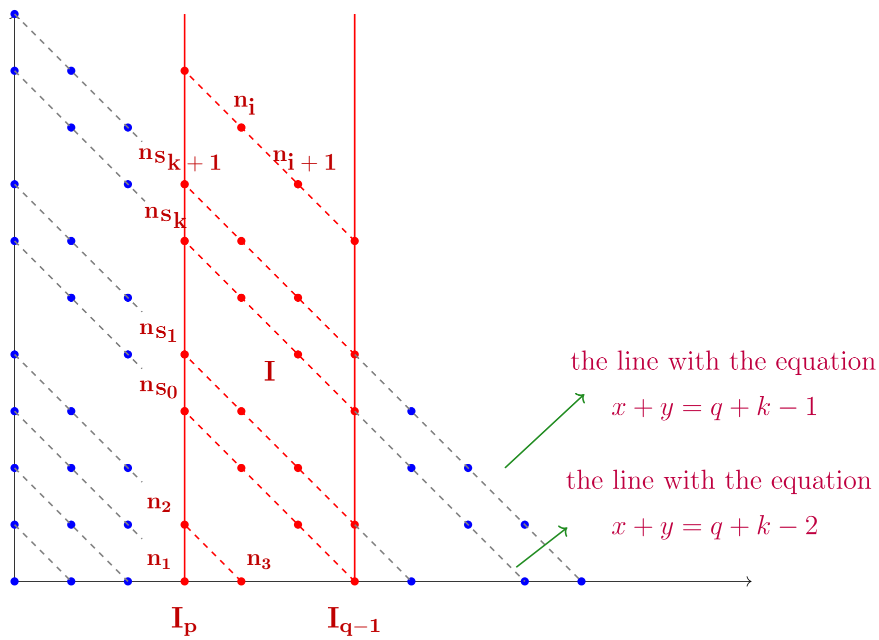
### 3.1. Kolmogorov diameters of an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$.

In this subsection, we calculate Kolmogorov diameters of an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$. In order to determine $n^{th}$-Kolmogorov diameter of a Köthe space $\mathcal{K}_\alpha$, we will rewrite the sequence $\left(\dfrac{a_{p,n}}{a_{q,n}}\right)_{n \in \mathbb{N}}$ in descending order. We know from Remark 2.5 that the $n^{th}$-Kolmogorov diameter of the space $\mathcal{K}_\alpha$ is the $n + 1^{th}$-term of this descending sequence.

Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ parameterized by an exponent sequence $\alpha$. Let us take a $p$, a $q > p$ and an $n \in I_s$, $s \in \mathbb{N}$. Then, we can write

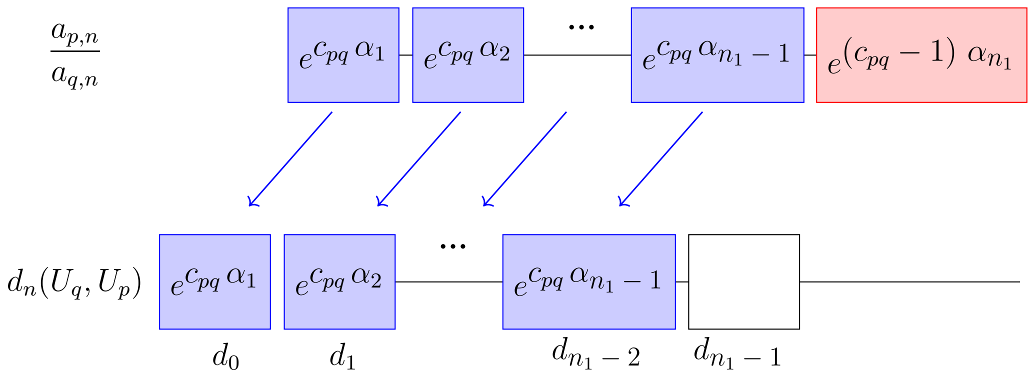
$$\frac{a_{p,n}}{a_{q,n}} = \begin{cases} e^{c_{pq}\,\alpha_n}, & s \geq q \text{ or } s < p \\[2ex] e^{\left(c_{pq} - 1\right)\alpha_n}, & p \leq s < q \end{cases}$$

where $c_{pq}$ is the negative number $-\dfrac{1}{p}+\dfrac{1}{q}$. We define the set $I=\bigcup_{p\le s<q} I_s$ with the elements $(n_i)_{i\in\mathbb{N}}$ ordered increasingly, namely, $n_i \le n_{i+1}$ for all $i\in\mathbb{N}$. We also denote the index of the element of $I_p$ on the line with the equation $x+y=q+k-2$ by $\boldsymbol{s_k}$ for each $k=0,1,2,\ldots$, as seen from the following diagram. Since every a line with the equation $x+y=q+k-2$ has $q-p$ elements of $I$, then $s_{(k+1)}-s_k=q-p$ for every $k=0,1,2,\ldots$.




Now we assume that the terms $e^{c_{pq}\,\alpha_m}$, $m\in\mathbb{N}-I$, are on the blue points and the terms $e^{(c_{pq}-1)\,\alpha_{n_i}}$, $n_i\in I$, are on the red points at this line. Before sorting the terms of the sequence $\left(\dfrac{a_{p,n}}{a_{q,n}}\right)_{n\in\mathbb{N}}$, we note that the terms of the sequences $\left(e^{c_{pq}\alpha_m}\right)_{m\in\mathbb{N}-I}$ and $\left(e^{(c_{pq}-1)\,\alpha_{n_i}}\right)_{i\in\mathbb{N}}$ have decreasing order in themselves.

At first, we take into account the part of $\left(\dfrac{a_{p,n}}{a_{q,n}}\right)_{n\in\mathbb{N}}$ including the first $n_1-1$ terms $e^{(c_{pq}-1)\,\alpha_{n_i}}$, $1\le i\le n-1$. Since $\alpha$ is increasing, this part has decreasing order and all terms in this part is greater than the terms corresponding to the elements of $I$. Then, having decreasing order, this part remains the same. However, we write this part by shifting to the left taking into account the zero indices for Kolmogorov diameter.




So, for every $0 \leq n \leq n_1 - 2$,

$$d_n(U_q, U_p) = e^{c_{pq}\alpha_{n+1}}.$$

In order to find the diameter $d_{n_1-1}(U_q, U_p)$, we will compare the term $e^{(c_{pq}-1)\alpha_{n_1}}$ with the terms $e^{c_{pq}\alpha_m}$, $m \in \mathbb{N} - I$, $m > n_1$, and the greatest term gives the diameter $d_{n_1-1}(U_q, U_p)$:

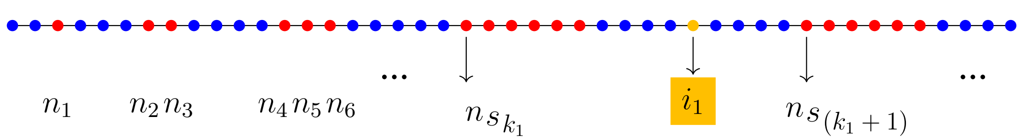
$$e^{(c_{pq}-1)\alpha_{n_1}} \leq e^{c_{pq}\alpha_m} \qquad \Leftrightarrow \qquad \alpha_m \leq A_{pq}\,\alpha_{n_1}.$$

where $A_{pq} = 1 + \dfrac{pq}{q-p}$. Then, the terms $e^{c_{pq}\alpha_m}$, $m \in \mathbb{N} - I$, $m > n_1$, satisfying $\alpha_m \leq A_{pq}\,\alpha_{n_1}$ is greater than the term $e^{(c_{pq}-1)\alpha_{n_1}}$. So we must write the terms $e^{c_{pq}\alpha_m}$, $m \in \mathbb{N} - I$, $m > n_1$, satisfying $\alpha_m \leq A_{pq}\,\alpha_{n_1}$ before the term $e^{(c_{pq}-1)\alpha_{n_1}}$ in decreasing order.

We call the greatest element $m \in \mathbb{N} - I$ satisfying $\alpha_m \leq A_{pq}\,\alpha_{n_1}$ as $\mathbf{i_1}$. As shown in the following diagram, we can assume that there exists a $k_1 > 0$ so that the inequality

$$n_{s_{k_1}} < i_1 < n_{s_{(k_1+1)}}$$

holds.




This means that the number of elements of $I$ which is less than $i_1$ is $\boldsymbol{s_{(k_1+1)} - 1}$. So, before the term $e^{(c_{pq}-1)\alpha_{n_1}}$, we will write $i_1 - [s_{(k_1+1)} - 1]$ many $e^{c_{pq}\alpha_m}$, $m \in \mathbb{N} - I$, $m \leq i_1$, terms in decreasing order. Furthermore, while writing these terms in decreasing order, every term $e^{(c_{pq}-1)\alpha_{n_a}}$, $1 \leq a \leq s_{(k_1+1)} - 1$ shifts to the right and every term $e^{c_{pq}\alpha_m}$, $m \in \mathbb{N} - I$, $m \leq i_1$, shifts to the left, as shown in Diagram 1.

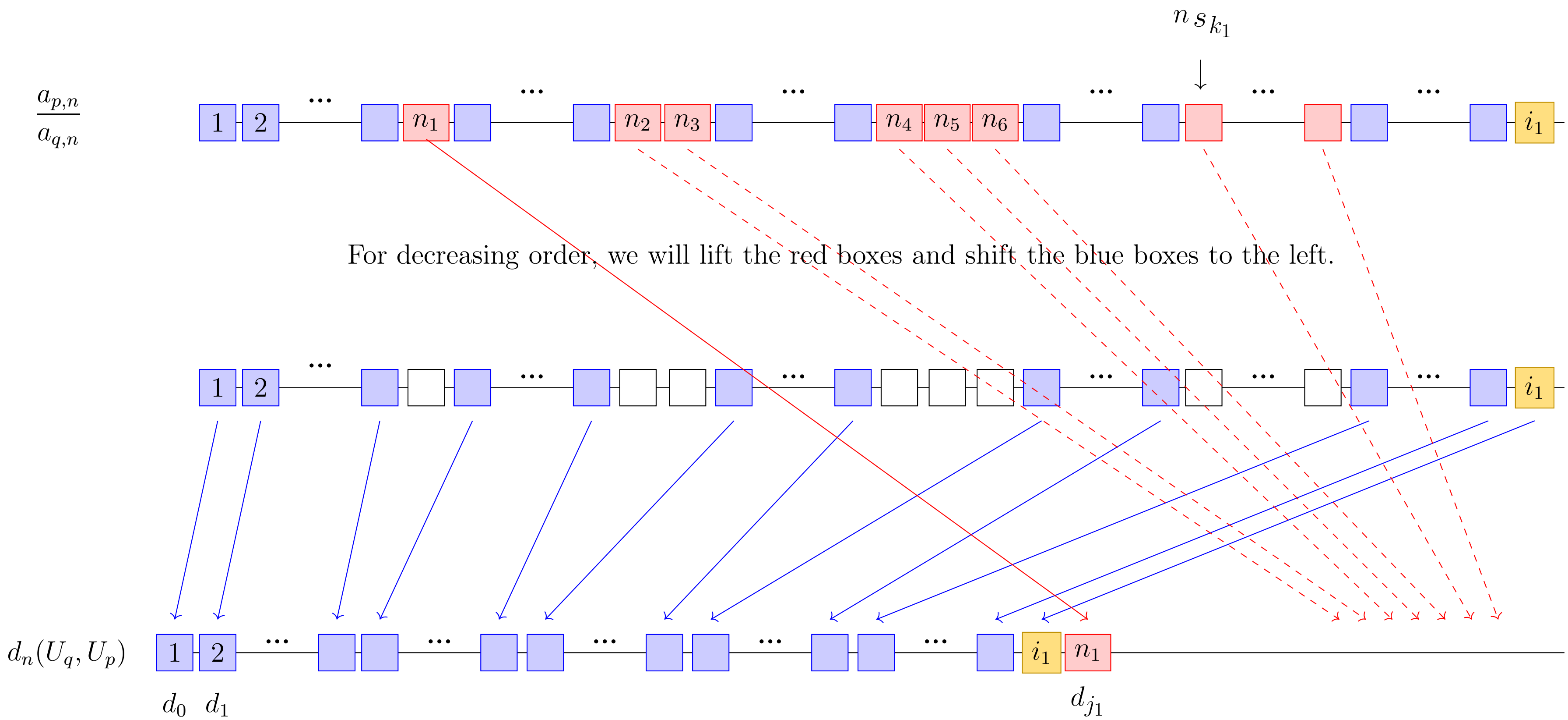

$n_{s_{k_1}}$
$\frac{a_{p,n}}{a_{q,n}}$
1
2
$n_1$
$n_2$
$n_3$
$n_4$
$n_5$
$n_6$
$i_1$
For decreasing order, we will lift the red boxes and shift the blue boxes to the left.
$d_n(U_q, U_p)$
$d_0$
$d_1$
$d_{j_1}$

**Diagram 1**

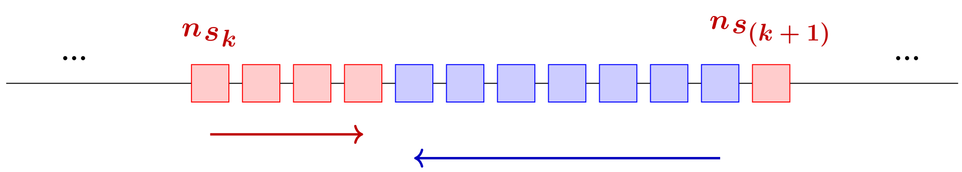
In order to find $n_1-1$-th Kolmogorov diameter, we shift the term corresponding to the first element $n_1$ of $I$. Considering also that we shift the terms to the left for $d_0(U_q,U_p)$, we find that for every $n_1-1\le n\le n_2-3$,

$$d_n(U_q,U_p)=e^{c_{pq}\alpha_{n+2}}.$$

So, we found the Kolmogorov diameters until the indices $n_2-2$. Now, we also shift the terms corresponding to the element $n_2$ and $n_3$ of $I$. Up till now, we shift the terms to the left four-indices, then we find that for every $n_2-2\le n\le n_4-5$

$$d_n(U_q,U_p)=e^{c_{pq}\alpha_{n+4}}.$$

We would like to point out that the endpoints of the intervals in which we determine Kolmogorov diameters are generally represented by the elements of $I_p$. Because the terms corresponding to the elements of $I$ that we shift to the right and the terms corresponding to the elements of $\mathbb{N}-I$ that we shift to the left are between the two elements of $I_p$, as seen in the following diagram.




Another significant point in writing the endpoints of the intervals in which we determine the diameters are to find out how many elements, the terms corresponding to the elements of $I$, we shift to the right.

We continue to calculate the diameters with this perspective. Let us assume that we replaced $n_{s_0}-[s_0+1]$ terms in decreasing order. In order to find $n_{s_0}-s_0$-th Kolmogorov diameter, we shift $s_1$ terms corresponding to the elements of I in total, for every $n_{s_0}-s_0\le n\le n_{s_1}-[s_1+1]$, we have

$$d_n(U_q,U_p)=e^{c_{pq}\alpha_{n+s_1}}.$$

Considering the terms that we shift to the right in each step, we can write for every $0\le k<k_1$ and for every $n_{s_k}-s_k\le n\le n_{s_{(k+1)}}-[s_{(k+1)}+1]$

$$d_n\left(U_q,U_p\right)=e^{c_{pq}\alpha_{n+s_{(k+1)}}}$$

and for all $n_{s_{k_1}}-s_{k_1}\le n\le i_1-s_{(k_1+1)}$

$$d_n\left(U_q,U_p\right)=e^{c_{pq}\alpha_{n+s_{(k_1+1)}}}.$$

Therefore, we shift $i_1-[s_{(k_1+1)}-1]$ many terms $e^{c_{pq}\alpha_m}$, $m\in\mathbb{N}-I$, $m\le i_1$ to left, namely, we sort all terms which is greater than $e^{(c_{pq}-1)\alpha_{n_1}}$. Hence, the term $e^{(c_{pq}-1)\alpha_{n_1}}$ is replaced at the indices $\boldsymbol{j_1} = i_1 - s_{(k_1+1)} + 1$, namely,

$$d_{j_1}(U_q, U_p) = e^{(c_{pq}-1)\alpha_{n_1}}.$$

Now assume that the first $a-1$, $(a \geq 2)$ terms corresponding to the elements of $I$ are placed in decreasing order. Before the term $e^{(c_{pq}-1)\alpha_{n_a}}$, we must write the terms $e^{c_{pq}\alpha_m}$, $m \in \mathbb{N}-I$ which is greater than $e^{(c_{pq}-1)\alpha_{n_a}}$, satisfying the inequality $\alpha_m \leq A_{pq}\alpha_{n_a}$ We call the greatest element of $m \in \mathbb{N}$ satisfying $\alpha_m \leq A_{pq}\alpha_{n_a}$ as $\boldsymbol{i_a}$. We can assume that there exists a $k_a \in \mathbb{N}$ so that

$$n_{s_{k_a}} < i_a < n_{s_{(k_a+1)}}.$$

This means that the number of elements of $I$ which is less than $i_a$ is $s_{(k_a+1)} - 1$. So, before the term $e^{(c_{pq}-1)\alpha_{n_a}}$, we write $i_a - (s_{(k_a+1)} - 1)$ many $e^{c_{pq}\alpha_m}$, $m \in \mathbb{N} - I$, $m \leq i_a$, terms in decreasing order. Since we assume that the first $a-1$ terms corresponding to the elements of $I$ is placed in decreasing order, then the term $e^{(c_{pq}-1)\alpha_{n_a}}$ are replaced at the indices $\boldsymbol{j_a} = i_a - s_{(k_a+1)} - 1 + a$, namely,

$$d_{j_a}(U_q, U_p) = e^{(c_{pq}-1)\alpha_{n_a}}.$$

Now, we determine Kolmogorov diameters between the indices $j_a$ and $j_{(a+1)}$ for every $a \geq 1$. Starting the index $j_a$, we must compare the term $e^{(c_{pq}-1)\alpha_{n_{a+1}}}$ with the terms $e^{c_{pq}\alpha_m}$, $m \in \mathbb{N}-I$ and for every $m \in \mathbb{N}-I$ satisfying $\alpha_m \leq A_{pq}\alpha_{n_{a+1}}$, we write the terms $e^{c_{pq}\alpha_m}$, before the term $e^{(c_{pq}-1)\alpha_{n_{a+1}}}$. Again, we call the largest element of $\mathbb{N}-I$ satisfying above inequality as $i_{(a+1)}$ for which there is $k_{(a+1)} \in \mathbb{N}$ satisfying

$$n_{s_{k_{(a+1)}}} < i_{(a+1)} < n_{s_{(k_{(a+1)}+1)}}.$$

Let us continue to decreasing order from $j_a + 1$:
For all $j_a + 1 \leq n \leq n_{s_{(k_a+1)}} - s_{(k_a+1)} + a - 1$

$$d_n(U_q, U_p) = e^{c_{pq}\alpha_{n + s_{(k_a+1)} - a}}.$$

If any, for every $k_a + 1 \leq k \leq k_{(a+1)} - 1$ and for every $n_{s_k} - s_k + a \leq n \leq n_{s_{(k+1)}} - s_{(k+1)} + a - 1$

$$d_n(U_q, U_p) = e^{c_{pq}\alpha_{n + s_{(k+1)} - a}}$$

and for every $n_{s_{k_{(a+1)}}} - s_{k_{(a+1)}} + a \leq n \leq i_{(a+1)} - s_{(k_{(a+1)}+1)} + a$

$$d_n(U_q, U_p) = e^{c_{pq}\alpha_{n + s_{(k_{(a+1)}+1)} - a}}.$$

We sort all terms which is greater than $e^{(c_{pq}-1)\alpha n_{a+1}}$. Then, the term $e^{(c_{pq}-1)\alpha n_{a+1}}$ is replaced at the indices $j_{(a+1)} = i_{(a+1)} - s_{(k_{(a+1)}+1)} + a + 1$, namely,

$$d_{j_{(a+1)}}(U_q, U_p) = e^{(c_{pq}-1)\alpha n_{(a+1)}}.$$

Hence, we determine all Kolmogorov diameters between the terms $e^{(c_{pq}-1)\alpha n_a}$ and $e^{(c_{pq}-1)\alpha n_{(a+1)}}$ for every $a \geq 1$.

Therefore, we can calculate all Kolmogorov diameters by following the above observation, and finally we can write:

1. Let $J := \{j_a : a \in \mathbb{N}\}$ where $j_a = i_a - s_{(k_a+1)} - 1 + a$. For all $a \in \mathbb{N}$,

$$\boxed{d_{j_a}(U_q, U_p) = e^{(c_{pq}-1)\alpha n_a}.}$$

2. For $a, k \in \mathbb{N}$, we define

$$I_{a,k} = \left[n s_k - s_k + a, n s_{(k+1)} - s_{(k+1)} + a - 1\right]$$

and

$$K = \bigcup_{a \in \mathbb{N}} \bigcup_{k_a+1 \leq k \leq k_{a+1}-1} I_{a,k}.$$

For every $n \in K$, there is an $a \in \mathbb{N}$ and a $k \in \mathbb{N}$ satisfying $k_a + 1 \leq k \leq k_{(a+1)} - 1$ such that

$$\boxed{d_n(U_q, U_p) = e^{c_{pq}\alpha n + s_{(k+1)} - a}.}$$

3. Let $L = \bigcup_{a \in \mathbb{N}} [\ j_a + 1,\ n\, s_{(k_a+1)} - s_{(k_a+1)} + a - 1\ ]$. For every $n \in L$, there is an $a \in \mathbb{N}$ such that

$$\boxed{d_n(U_q, U_p) = e^{c_{pq}\alpha n + s_{(k_a+1)} - a}.}$$

4. Let $M = \bigcup_{a \in \mathbb{N}} \left[n s_{k_a} - s_{k_a} + a - 1, j_a - 1\right]$. For every $n \in M$, there is an $a \in \mathbb{N}$ such that

$$\boxed{d_n(U_q, U_p) = e^{c_{pq}\alpha n + s_{(k_a+1)} - (a-1)}}.$$

All Kolmogorov diameters in the light of above observation are found since $\mathbb{N} = \{0, 1, ..., n_1 - 2\} \cup J \cup K \cup L \cup M$. This completes the determination of the diameters.

Now, we give an estimation for Kolmogorov diameters of an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ which is parameterized by $\alpha$.

**Theorem 3.4.** *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ with the parameter $\alpha$. For every $p$, $q > p$ there exists a $N \in \mathbb{N}$ such that*

$$e^{c_{pq}\alpha_{4n}} \leq d_n(U_q, U_p) \leq e^{c_{pq}\alpha_n} \tag{3.4}$$

*for every $n \geq N$.*

*Proof.* Let $p \in \mathbb{N}$ and $q > p$. Above we obtained Kolmogorov diameters $d_n(U_q, U_p)$ on each subsets $\{0, 1, ..., n_1 - 1\}$, $J$, $K$, $L$ and $M$ of $\mathbb{N}$. We will show that the inequality 3.4 holds for sufficiently large elements of each subsets $J$, $K$, $L$, and $M$ of $\mathbb{N}$.

Primarily, we will show that $2j_a > i_a$ for sufficiently large $a \in \mathbb{N}$. We know that for every $i_a$ there exists a $k_a \in \mathbb{N}$ satisfying $n_{s_{k_a}} < i_a < n_{s_{(k_a+1)}}$. Since $n_{s_{k_a}}$ is on the line which has the equation $x+y = q+k_a-2$, the first element of $I_{q+k_a-2}$ is less than $n_{s_{k_a}}$, we can write

$$n_{s_{k_a}} \geq \frac{(q + k_a - 2)(q + k_a - 1)}{2} = \frac{(q-2)(q-1)}{2} + (q-1)k_a + \frac{k_a(k_a-1)}{2}.$$

Since $\lim_{a \to +\infty} k_a = +\infty$, we can assume that $\frac{k_a(k_a-1)}{4} \geq (k_a+1)(q-p)$ and $\frac{(q-1)k_a}{2} \geq s_0$ for sufficiently large $a \in \mathbb{N}$. Hence we can write

$$\frac{i_a}{2} \geq \frac{n_{s_{k_a}}}{2} \geq s_0 + (k_a+1)(q-p) = s_{(k_a+1)}$$

and we find

$$j_a = i_a - s_{(k_a+1)} + a > i_a - \frac{i_a}{2} = \frac{i_a}{2} \quad \Rightarrow \quad 2j_a > i_a.$$

Now, we will show that the inequality 3.4 is satisfied for a sufficiently large element of $J$. Let take an $a \in \mathbb{N}$ satisfiying $2j_a > i_a$. We know that $i_a$ is the greatest element of $m \in \mathbb{N} - I$ satisfying $e^{(c_{pq}-1)\alpha_{n_a}} \leq e^{c_{pq}\alpha_m}$, then we can write

$$e^{c_{pq}\alpha_k} < e^{(c_{pq}-1)\alpha_{n_a}}$$

for every $k > i_a$, $k \in \mathbb{N} - I$. If $2j_a \in \mathbb{N} - I$, then

$$e^{c_{pq}\alpha_{4j_a}} \leq e^{c_{pq}\alpha_{2j_a}} < e^{(c_{pq}-1)\alpha_{n_a}}.$$

If $2j_a \in I$, then $2j_a + (q-p) \in \mathbb{N} - I$ and $2j_a + (q-p) \leq 4j_a$ is satisfied for a sufficiently large $a$ and we find

$$e^{c_{pq}\alpha_{4j_a}} \leq e^{c_{pq}\alpha_{2j_a+(q-p)}} < e^{(c_{pq}-1)\alpha_{n_a}}.$$

Also, we know that $i_a \geq j_a$ for every $a \in \mathbb{N}$, thus we can write

$$d_{j_a}(U_q, U_p) = e^{(c_{pq}-1)\alpha_{n_a}} \leq e^{c_{pq}\alpha_{i_a}} \leq e^{c_{pq}\alpha_{j_a}}.$$

The above inequalites give us that

$$e^{c_{pq}\alpha_{4ja}} \leq d_{j_a}(U_q, U_p) = e^{(c_{pq}-1)\alpha_{n_a}} \leq e^{c_{pq}\alpha_{j_a}}.$$

Then, the inequality 3.4 is satisfied for sufficiently large element of $J$.

We now prove that the inequality 3.4 is satisfied for sufficiently large elements of $K$, $L$ and $M$. In order to see this, we first show that

$$n_{s_k} \geq 2s_k$$

for sufficently large $k \in \mathbb{N}$. We know that $n_{s_k}$ is on the line which has equation $x + y = q + k - 2$ for every $k = 0, 1....$ Since the first element of $I_{q+k_a-2}$ is less than $n_{s_k}$, then we can write

$$n_{s_k} \geq \frac{(q+k-2)(q+k-1)}{2} = \frac{(q-2)(q-1)}{2} + (q-1)k + \frac{k(k-1)}{2}.$$

The inequalities

$$\frac{k.(k-1)}{4} \geq k(q-p) \qquad \text{and} \qquad (q-1)k \geq 2(s_0+1)$$

hold for a sufficiently large $k$. Then we find

$$n_{s_k} \geq 2(s_0 + k(q-p) + 1) = 2s_k$$

for a sufficiently large $k \in \mathbb{N}$.

Now we show that the inequality 3.4 is satisfied for sufficiently large element of $K$. Let take an $n \in K$. Then, there exist a $a \in \mathbb{N}$ and a $k \in \mathbb{N}$ satisfying $k_a + 1 \leq k \leq k_{(a+1)} - 1$ such that $n_{s_k} - s_k + a \leq n \leq n_{s_{(k+1)}} - s_{(k+1)} + a - 1$ and

$$d_n(U_q, U_p) = e^{c_{pq}\alpha_{n+s_{(k+1)}-a}}.$$

Since $n_{s_k} \geq 2s_k$ for a sufficiently large $k \in \mathbb{N}$ and $s_{(k+1)} - s_k = q - p$ for all $k \in \mathbb{N}$, we can write

$$s_k \leq n_{s_k} - s_k + a \leq n \quad \Rightarrow \quad n + s_{(k+1)} - a \leq 2n.$$

for sufficiently large $a$. Then, we obtain

$$d_n(U_q, U_p) = e^{c_{pq}\alpha_{n+s_{(k+1)}-a}} \geq e^{c_{pq}\alpha_{2n}} \geq e^{c_{pq}\alpha_{4n}}$$

and always we have

$$d_n(U_q, U_p) = e^{c_{pq}\alpha_{n+s_{(k+1)}-a}} \leq e^{c_{pq}\alpha_n}$$

since $\alpha$ is increasing. Therefore, the inequality 3.4 is satisfied for sufficiently large elemets of $K$.

Now, we will show that the inequality 3.4 is satisfied for a sufficiently large element of $L$. Let us take a $n \in L$. Then, there is an $a \in \mathbb{N}$ such that

$$j_a + 1 \leq n \leq n_{s_{(k_a+1)}} - s_{(k_a+1)} + a - 1$$

and

$$d_n\left(U_q, U_p\right) = e^{c_{pq}\alpha_{n + s_{(k_a+1)} - a}}.$$

Since $s_{k_a} \leq ns_{k_a} - s_{k_a} + a \leq j_a + 1 \leq n$ and $n + s_{(k_a+1)} - a \leq 2n$ for a sufficiently large $n$, then we find

$$d_n\left(U_q, U_p\right) = e^{c_{pq}\alpha_{n + s_{(k_a+1)} - a}} \geq e^{c_{pq}\alpha_{2n}} \geq e^{c_{pq}\alpha_{4n}},$$

and always we have

$$d_n\left(U_q, U_p\right) = e^{c_{pq}\alpha_{n + s_{(k_a+1)} - a}} \leq e^{c_{pq}\alpha_n}$$

since $\alpha$ is increasing. Therefore, the inequality 3.4 is satisfied for sufficiently large element of $L$.

Now we will show that the inequality 3.4 is satisfied for a sufficiently large element of $M$. If $n \in M$, then there is an $a \in \mathbb{N}$

$$ns_{k_a} - s_{(k_a+1)} + a \leq n \leq j_a - 1$$

and

$$d_n\left(U_q, U_p\right) = e^{c_{pq}\alpha_{n + s_{(k_a+1)} - (a-1)}}.$$

Again we can write $s_{k_a} \leq ns_{k_a} - s_{k_a} + a \leq n$ and $ns_{k_a} + s_{k_a} - a + 1 \leq 2n$ for a sufficiently large $a$. Hence we find

$$d_n\left(U_q, U_p\right) = e^{c_{pq}\alpha_{n + s_{(k_a+1)} - (a-1)}} \geq e^{c_{pq}\alpha_{2n}} \geq e^{c_{pq}\alpha_{4n}}$$

and always we have

$$d_n\left(U_q, U_p\right) = e^{c_{pq}\alpha_{n + s_{(k_a+1)} - (a-1)}} \leq e^{c_{pq}\alpha_n}$$

since $\alpha$ is increasing. Therefore, the inequality 3.4 is satisfied for a sufficiently large element of $M$. This completes the proof. □

### 3.2. The diametral dimension and the approximate diametral dimension of an element of the family $\mathcal{K}$ parameterized by a sequence $\boldsymbol{\alpha}$.

As a consequence of Theorem 3.4, we will compute the diametral dimension and the approximate diametral dimension of an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ which is parameterized by a stable sequence $\alpha$.

**Corollary 3.5.** *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ which is parameterized by a stable sequence $\alpha$. Then, $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1(\alpha_n))$ and $\delta(\mathcal{K}_\alpha) = \delta(\Lambda_1(\alpha_n))$.*

*Proof.* From Theorem 3.4, we have

$$\Delta(\Lambda_1(\alpha_n)) \subseteq \Delta(\mathcal{K}_\alpha) \subseteq \Delta(\Lambda_1(\alpha_{4n}))$$

and

$$\delta(\Lambda_1(\alpha_{4n})) \subseteq \delta(\mathcal{K}_\alpha) \subseteq \delta(\Lambda_1(\alpha_n)).$$

On the other hand, $\Lambda_1(\alpha_n) \cong \Lambda_1(\alpha_{4n})$ since $\alpha$ is stable. Then $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1(\alpha_n))$ and $\delta(\mathcal{K}_\alpha) = \delta(\Lambda_1(\alpha_n))$. $\square$

Now we will prove that $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1(\alpha_{n+1}))$ and $\delta(\mathcal{K}_\alpha) \neq \delta(\Lambda_1(\alpha_{n+1}))$ for an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ which is parameterized by an unstable sequence $\alpha$. Besides, we will show that all regular elements of the family $\mathcal{K}$ are parameterized by an unstable sequence $\alpha$.

**Proposition 3.6.** *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ which is parameterized by an unstable sequence $\alpha$. Then, $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1(\alpha_{n+1}))$.*

*Proof.* We can calculate Kolmogorov diameters as in the previous determined for every $p$ and $q > p$. Since $\alpha$ is unstable, then there exists an $a_0 \in \mathbb{N}$ such that for all $a \geq a_0$, there is no $m > n_a$, $m \in \mathbb{N}$ satisfying $\alpha_m \leq A_{pq}\,\alpha_{n_a}$. Now, we examine closely the indices replaced the term $e^{(c_{pq}-1)\alpha_{n_{a_0}}}$. We know that

$$d_{j_{(a_0-1)}}(U_q, U_p) = e^{(c_{pq}-1)\alpha_{n_{(a_0-1)}}}$$

where $j_{(a_0-1)} = i_{(a_0-1)} - s_{(a_0-1)} + a_0 - 2$. Since $\alpha_{i_{(a_0-1)}} \leq A_{pq}\,\alpha_{n_{(a_0-1)}}$ and there is no $m > n_{a_0}$ satisfying $\alpha_m \leq A_{pq}\,\alpha_{n_{a_0}}$, then we find $i_{(a_0-1)} < n_{a_0}$. This gives that for all $j_{(a_0-1)} \leq n \leq n_{a_0} - 2$,

$$d_n(U_q, U_p) = e^{c_{pq}\,\alpha_{n+1}}.$$

Besides, we obtain that the sequence $\left(\dfrac{a_{p,n}}{a_{q,n}}\right)_{n\in\mathbb{N}}$ has decreasing order starting from the indices $j_{(a_0-1)} + 1$, since for every $a \geq a_0$, there is no $n > n_{a_0}$ satisfying $\alpha_n \leq A_{pq}\,\alpha_{n_a}$. Then, we have for all $a \geq a_0$

$$d_{n_a - 1}(U_q, U_p) = e^{(c_{pq}-1)\alpha_{n_a}}$$

and for all $m \geq j_{(a_0-1)}$, $m \in \mathbb{N} - I$

$$d_m(U_q, U_p) = e^{c_{pq}\,\alpha_{m+1}}.$$

Since $d_n(U_q, U_p) \leq e^{c_{pq}\,\alpha_{n+1}}$ for every $n \in \mathbb{N}$, then we find $\Delta(\mathcal{K}_\alpha) \supseteq \Delta(\Lambda_1(\alpha_{n+1}))$.

For the other direction, let us take a sequence $(x_n)_{n\in\mathbb{N}} \in \Delta(\mathcal{K}_\alpha)$, an $\varepsilon > 0$ and a $p \in \mathbb{N}$ satisfying $\dfrac{1}{p} < \varepsilon$. We will show that

$$\sup_{n\in\mathbb{N}} |x_n|\, e^{-\varepsilon\,\alpha_{n+1}} < +\infty.$$

Since $(x_n)_{n\in\mathbb{N}} \in \Delta(\mathcal{K}_\alpha)$, there exist a $q > p$ and $M_1 > 0$ satisfying

$$\sup_{n\in\mathbb{N}} |x_n|\, d_n\,(U_p, U_q) < M_1.$$

Let us define $I = \bigcup_{p \le s < q} I_s$. For sufficiently large $n \in \mathbb{N} - I$, we can write

$$|x_n|\, e^{-\varepsilon\alpha_{n+1}} \le |x_n|\, d_n\,(U_q, U_p) = e^{c_{pq}\alpha_{n+1}} \le M_1$$

since $c_{pq} \ge -\varepsilon$. Therefore, the sequence $|x_n|\, e^{-\varepsilon\alpha_{n+1}}$ is bounded on the set $\mathbb{N} - I$. If we show that $|x_n|\, e^{-\varepsilon\alpha_{n+1}}$ is also bounded on $I$, then we will find that $(x_n)_{n\in\mathbb{N}} \in \Delta(\Lambda_1\,(\alpha_{n+1}))$. Let take another $p_0 > q$, then there exist a $q_0$ and $M_2 > 0$ such that

$$\sup_{n\in\mathbb{N}} |x_n|\, d_n\left(U_{q_0}, U_{p_0}\right) < M_2.$$

Let us define $J = \bigcup_{p_0 \le s < q_0} I_s$. Since $c_{p_0,\,q_0} \ge -\varepsilon$, we find

$$|x_n|\, e^{-\varepsilon\alpha_{n+1}} \le |x_n|\, d_n\left(U_{q_0}, U_{p_0}\right) = e^{c_{p_0,\,q_0}\alpha_{n+1}} \le M_2$$

for sufficently large $n+1 \in \mathbb{N} - J$. Also, it is easy to see that $I \subset \mathbb{N} - J$. Then, the above inequalities give us that

$$|x_n|\, e^{-\varepsilon\alpha_{n+1}} \le M_2.$$

for all $n \in I$. Hence, the sequence $|x_n|\, e^{-\varepsilon\alpha_{n+1}}$ is also bounded on $I$. Therefore, we find

$$\sup_{n\in\mathbb{N}} |x_n|\, e^{-\varepsilon\alpha_{n+1}} < +\infty$$

and $(x_n)_{n\in\mathbb{N}} \in \Delta(\Lambda_1\,(\alpha_{n+1}))$. This says that $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1\,(\alpha_{n+1}))$. □

**Proposition 3.7.** *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ which is parameterized by an unstable sequence $\alpha$. Then, $\delta(\mathcal{K}_\alpha) \neq \delta(\Lambda_1\,(\alpha_{n+1}))$.*

*Proof.* In the proof of the previous proposition, we show that if $\alpha$ is unstable, then for all $p \in \mathbb{N}$ and $q > p$, there is a $a_0 \in \mathbb{N}$ such that for all $a \ge a_0$

$$d_{n_a - 1}(U_q, U_p) = e^{(c_{pq} - 1)\alpha_{n_a}},$$

so the last equality holds except for finitely many numbers of elements of $I$. Then we have

$$\frac{\varepsilon_{n_a - 1}\,(p, q)}{\alpha_{n_a}} = 1 - c_{p,q}$$

and

$$\limsup_{a\in\mathbb{N}} \frac{\varepsilon_{n_a-1}(p,q)}{\alpha_{n_a}} = 1 - c_{p,q} \qquad \Rightarrow \qquad \inf_p \sup_q \limsup_{n\in\mathbb{N}} \frac{\varepsilon_n(p,q)}{\alpha_{n+1}} > 0.$$

By Proposition 2.13, we have $\delta(\mathcal{K}_\alpha) \neq \delta(\Lambda_1(\alpha_{n+1}))$. □

**Remark 3.8.** *Proposition 3.6 and Proposition 3.7 shows that Question 1.2 has a negative answer for the elements of the family $\mathcal{K}$ which is parametrized by an unstable exponent sequence.*

Now, we will show that all regular elements of the family $\mathcal{K}$ are parameterized by an unstable sequence $\alpha$.

Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ parameterized by an exponent sequence $\alpha$ and $n \in I_s$, $s \in \mathbb{N}$. Then, there exist two cases for $n+1$: $n+1 \in I_{s+1}$ or $n+1 \in I_1$.

We assume $n+1 \in I_{s+1}$: For this case, $n+1 \geq \dfrac{(s+1)(s+2)}{2} \geq s+1$.

**i)** For $k+1 \leq s$, we have $a_{k,n} = e^{-\frac{1}{k}\alpha_n}$, $a_{k+1,n} = e^{-\frac{1}{k+1}\alpha_n}$, $a_{k,n+1} = e^{-\frac{1}{k}\alpha_{n+1}}$ $a_{k+1,n+1} = e^{-\frac{1}{k+1}\alpha_{n+1}}$. Since $\alpha$ is increasing, the inequality

$$\frac{a_{k+1,n}}{a_{k,n}} = e^{\left(\frac{1}{k}-\frac{1}{k+1}\right)\alpha_n} \leq e^{\left(\frac{1}{k}-\frac{1}{k+1}\right)\alpha_{n+1}} = \frac{a_{k+1,n+1}}{a_{k,n+1}}$$

holds in this case.

**ii)** For $k \geq s+1$, we have $a_{k,n} = e^{\left(-\frac{1}{k}+1\right)\alpha_n}$, $a_{k+1,n} = e^{\left(-\frac{1}{k+1}+1\right)\alpha_n}$, $a_{k,n+1} = e^{\left(-\frac{1}{k}+1\right)\alpha_{n+1}}$, $a_{k+1,n+1} = e^{-\frac{1}{k+1}\alpha_{n+1}}$. Since $\alpha$ is increasing, the inequality

$$\frac{a_{k+1,n}}{a_{k,n}} = e^{\left(\frac{1}{k}-\frac{1}{k+1}\right)\alpha_n} \leq e^{\left(\frac{1}{k}-\frac{1}{k+1}\right)\alpha_{n+1}} = \frac{a_{k+1,n+1}}{a_{k,n+1}}$$

holds in this case.

**iii)** For $k = s$, we have $a_{k,n} = e^{-\frac{1}{k}\alpha_n}$, $a_{k+1,n} = e^{\left(-\frac{1}{k+1}+1\right)\alpha_n}$, $a_{k,n+1} = e^{-\frac{1}{k}\alpha_{n+1}}$, $a_{k+1,n+1} = e^{-\frac{1}{k+1}\alpha_{n+1}}$. Then, these give that

$$\frac{a_{k+1,n}}{a_{k,n}} = e^{\left(\frac{1}{k}-\frac{1}{k+1}+1\right)\alpha_n} \qquad \text{and} \qquad \frac{a_{k+1,n+1}}{a_{k,n+1}} = e^{\left(\frac{1}{k}-\frac{1}{k+1}\right)\alpha_{n+1}}.$$

In this case, the regularity condition $\dfrac{a_{k+1,n}}{a_{k,n}} \leq \dfrac{a_{k+1,n+1}}{a_{k,n+1}}$ is equivalent to the following inequality:

$$(1 + k(k+1))\alpha_n \leq \alpha_{n+1} \qquad \forall n \in I_k,\ k \in \mathbb{N} \tag{3.5}$$

The similiar observation can be given by following the same step for the case $n+1 \in I_1$. Then, we have a regularity condition for a Köthe space $\mathcal{K}_\alpha$:

**Proposition 3.9.** *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ parameterized by the sequence $\alpha$. Then, $\mathcal{K}_\alpha$ is regular if and only if the inequality*

$$(1 + s(s+1))\alpha_n \leq \alpha_{n+1}$$

*is satisfied for all $n \in I_s$ and $s \in \mathbb{N}$.*

We also note that the sequence $(\alpha_n)_{n\in\mathbb{N}} = \left(\prod_{i=0}^{n-1}(1 + i(i+1))\right)_{n\in\mathbb{N}}$ satisfies the condition of Proposition 3.9 since

$$\frac{\alpha_{n+1}}{\alpha_n} = (1 + n(n+1)) \geq (1 + s(s+1))$$

for all $n \in I_s$, $s \in \mathbb{N}$.

As a consequence of Proposition 3.9, we obtain the following result:

**Corollary 3.10.** *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ parameterized by the sequence $\alpha$. If $\mathcal{K}_\alpha$ is regular, then the sequence $\alpha$ is unstable.*

*Proof.* Let $\mathcal{K}_\alpha$ be a regular Köthe space generated by the matrix $(a_{k,n})_{k,n\in\mathbb{N}}$ given in 3.1 and assume $\alpha$ is not unstable, that is, $\lim\limits_{n\to\infty} \dfrac{\alpha_{n+1}}{\alpha_n} \neq +\infty$. Then, there exist a $M > 0$ and a non-decreasing sequence $(n_k)_{k\in\mathbb{N}}$ so that $\sup\limits_{k\in\mathbb{N}} \dfrac{\alpha_{n_k+1}}{\alpha_{n_k}} < M$. Since $(n_k)_{k\in\mathbb{N}}$ is non-decrasing and $\mathcal{K}_\alpha$ is regular, we can write

$$\frac{\alpha_{k+1}}{\alpha_k} \leq \frac{\alpha_{n_k+1}}{\alpha_{n_k}} \leq M$$

for all $k \in \mathbb{N}$ and from Proposition 3.9, we find that

$$(1 + s(s+1)) \leq \frac{\alpha_{k+1}}{\alpha_k} \leq M$$

for all $k \in I_s$, $s \in \mathbb{N}$. This is a contradiction, therefore $\alpha$ must be unstable, as desired. $\square$

**Remark 3.11.** *Being unstable is not sufficient for regularity of Köthe space $\mathcal{K}_\alpha$. For instance, the sequence $(\alpha_n)_{n\in\mathbb{N}} = (n!)_{n\in\mathbb{N}}$ does not satisfy the condition of Proposition 3.9. Indeed, for every $s \in \mathbb{N}$, $n = \dfrac{s(s+1)}{2} \in I_s$ and*

$$\frac{\alpha_{n+1}}{\alpha_n} = n + 1 = 1 + \frac{s(s+1)}{2} < 1 + s(s+1).$$

**Remark 3.12.** *As a corollary of Proposition 3.6, Proposition 3.7 and Corollary 3.10, we can obtain that* $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1\,(\alpha_{n+1}))$ *and* $\delta(\mathcal{K}_\alpha) \neq \delta(\Lambda_1\,(\alpha_{n+1}))$ *for a regular element* $\mathcal{K}_\alpha$ *of the family* $\mathcal{K}$ *which is parameterized by an exponent sequence* $\alpha$*.*

## 4. Some Results Obtained with the Family $\mathcal{K}$

In this section, we compile some additional information for the family $\mathcal{K}$. We have shown that an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ which is parametrized by an unstable sequence $\alpha$ constitutes a counterexample to Question 1.2. An element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ which is parametrized by an unstable sequence $\alpha$ is crucial for Question 1.1, as well:

**Theorem 4.1.** *There exists a nuclear Fréchet space* $E$ *with the properties* $\underline{DN}$ *and* $\Omega$ *satisfying* $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$*, for its associated exponent sequence* $\varepsilon$*, with the property that there is no subspace of* $E$ *which is isomorphic to* $\Lambda_1(\varepsilon)$*.*

*Proof.* Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ which is parametrized by an unstable sequence $\alpha$. We proved that $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1(\alpha_{n+1}))$ in Proposition 3.6. Therefore, the sequence $(\alpha_{n+1})_{n\in\mathbb{N}}$ is the associated exponent sequence of $\mathcal{K}_\alpha$. Assume that there exists a subspace of $\mathcal{K}_\alpha$ which is isomorphic to $\Lambda_1(\alpha_{n+1})$. This gives us that $\delta(\Lambda_1(\alpha_{n+1})) \subseteq \delta(\mathcal{K}_\alpha)$ by Proposition 2.1. Since always $\delta(\mathcal{K}_\alpha) \subseteq \delta(\Lambda_1(\alpha_{n+1}))$, we conclude that $\delta(\mathcal{K}_\alpha) = \delta(\Lambda_1(\alpha_{n+1}))$. But this is a contradiction since we showed that $\delta(\mathcal{K}_\alpha) \neq \delta(\Lambda_1(\alpha_{n+1}))$ in Proposition 3.7. Hence, there is no subspace of $\mathcal{K}_\alpha$ which is isomorphic to $\Lambda_1(\alpha_{n+1})$. □

**Remark 4.2.** *The above theorem indicates that Question 1.1 has a negative answer. It is worth mentioning that we can find even a nuclear regular Köthe space with the properties listed in Theorem 4.1.*

In [5], we gave conditions confirming an affirmative answer for Question 1.2. First result was related to the topology on diametral dimension of a nuclear Fréchet space. The diametral dimension

$$\begin{aligned}\Delta(E) &= \left\{(t_n)_{n\in\mathbb{N}} : \forall\, p \in \mathbb{N}\ \exists q > p\ \lim_{n\to\infty} t_n d_n(U_q, U_p) = 0\right\}\\ &= \bigcap_{p\in\mathbb{N}} \bigcup_{q>p} \Delta(U_q, U_p)\end{aligned}$$

is the projective limit of inductive limits of Banach spaces $\Delta\left(U_q, U_p\right)$ with the norm $\|(t_n)_n\| = \sup\limits_{n\in\mathbb{N}} |t_n|\, d_n(U_q, U_p)$. Hence $\Delta(E)$ is a topological vector space with respect to that topology which will be called *the canonical topology.*

**Theorem 4.3.** *Let $E$ be a nuclear Fréchet space with properties $\underline{DN}$ and $\Omega$ and $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$ be the associated exponent sequence of $E$. If $\Delta\left(E\right)$, with the canonical topology, is barrelled, then $\Delta\left(E\right) = \Delta\left(\Lambda_1\left(\varepsilon\right)\right)$ if and only if $\delta\left(E\right) = \delta\left(\Lambda_1\left(\varepsilon\right)\right)$.*

*Proof.* [5, Theorem 4.2] □

Therefore, we obtain the following:

**Proposition 4.4.** *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ parameterized by an unstable sequence $\alpha$. Then $\Delta(\mathcal{K}_\alpha)$, with the canonical topology, is neither barrelled nor ultrabornological.*

We actually wanted the barrelledness in [5, Theorem 4.2] to be able to use a closed graph type theorem, [7, Theorem 5, Pg. 40] which says that a linear map $f$ from a barrelled space $X$ into a Fréchet space $Y$ is continuous provided that the graph of f is closed in $X \times Y$. Since $\delta(\mathcal{K}_\alpha) \neq \delta(\Lambda_1(\alpha_{n+1}))$ and $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1(\alpha_{n+1}))$, the technique used in the proof of [5, Theorem 4.2] is not valid for an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ parameterized by an unstable sequence $\alpha$. Hence, this gives us that the identity mapping from $\Delta(\mathcal{K}_\alpha)$ into $\Lambda_1(\alpha_{n+1})$ is not continuous although it has a closed graph:

**Theorem 4.5.** *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ parameterized by an unstable sequence $\alpha$. Then $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1(\alpha_{n+1}))$ and the identity map from $\Delta(\mathcal{K}_\alpha)$ into $\Lambda_1(\alpha_{n+1})$ is not continuous although it has a closed graph.*

In [14], T. Terzioğlu defined the notion *prominent bounded subset* in order to show that the diametral dimension of some Fréchet spaces is determined by a single bounded set:

**Definition 4.6.** *Let $E$ be a Fréchet space. A bounded set $B$ is said to **prominent** if*

$$\Delta(E) = \left\{ (x_n)_{n\in\mathbb{N}} \;:\; \lim_{n\to+\infty} x_n\, d_n\left(B, U_p\right) = 0 \;\;\forall\, p \right\}.$$

The existence of a prominent bounded subset in the nuclear Fréchet space E plays a decisive role for the affirmative answer of Question 1.2.

**Theorem 4.7.** *Let $E$ be a nuclear Fréchet space with the properties $\underline{DN}$ and $\Omega$ and $\varepsilon$ the associated exponent sequence. $\delta(E) = \delta(\Lambda_1(\varepsilon))$ if and only if $E$ has a prominent bounded set and $\Delta(E) = \Lambda_1(\varepsilon)$.*

*Proof.* [5, Theorem 4.8] □

Obviously, this condition is not valid for an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ which is parameterized by an unstable sequence $\alpha$ since $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1(\alpha_{n+1})$ and $\delta(\mathcal{K}_\alpha) \neq \delta(\Lambda_1(\alpha_{n+1}))$.

**Theorem 4.8.** *There exists a nuclear Fréchet space $E$ with the properties $\underline{DN}$ and $\Omega$ satisfying $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$ for its associated exponent sequence $\varepsilon$ such that there is no prominent bounded set of $E$.*

**Remark 4.9.** *It is worth to note that as a consequence of Theorem 4.7 and Corollary 3.5, an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ parameterized by a $\underline{\text{stable}}$ sequence $\alpha$ has a prominent bounded subset.*

A nuclear Fréchet space $E$ with an increasing sequence of seminorms $(\|.\|_k)_{k\in\mathbb{N}}$ is called *tame* if there exists an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$, such that for every continuous linear operator $T : E \to E$ there exists a $n_0 \in \mathbb{N}$ and $C > 0$ so that

$$\|T(x)\|_k \leq C \|x\|_{\sigma(k)} \qquad \forall\, x \in E.$$

In [1, Theorem 2.3], A. Aytuna proved that a nuclear Fréchet space $E$ with the properties $\underline{DN}$ and $\Omega$ and stable associated exponent sequence $\varepsilon$ is isomorphic to a power series space of finite type if and only if $E$ is tame and $\delta(E) = \delta(\Lambda_1(\varepsilon))$. As a consequence of this result and Remark 4.9, we have the following:

**Proposition 4.10.** *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ parameterized by a stable sequence $\alpha$. Then, $\mathcal{K}_\alpha$ is not tame.*

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